ME 132

Feedback Control Systems

Linearization

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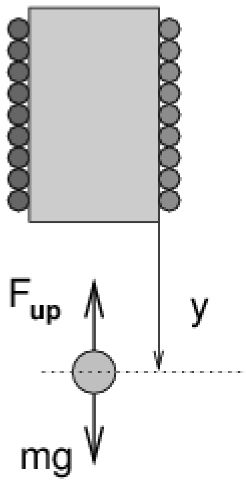
Review: examples of nonlinear systems (Section 2.9)

Magnetically Suspended Ball



$$F_{up} = -\frac{cu^2}{y^2}$$

u the current injected into the magnet coils

y the position of the ball

Newton's law gives

$$m\ddot{y} = mg - \frac{cu^2}{y^2}$$

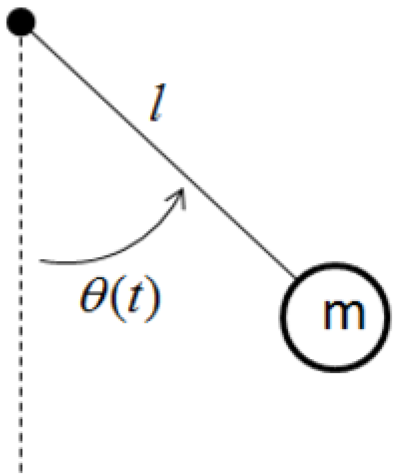
Define the states: $x_1 = y, x_2 = \dot{y}$

State-space model:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2, u) = g - \frac{cu^2}{mx_1^2} = 10 - \left(\frac{u}{3.87x_1}\right)^2 \\ y &= h(x_1, x_2, u) = x_1 \end{aligned}$$

Review: examples of nonlinear systems (Section 2.9)

Pendulum



Input: $u(t) = T_c(t)$, torque to the pivot point

Output: $y(t) = \theta(t)$, angle of the pendulum

Moment of inertia about the pivot point: $I = ml^2$

Rigid body dynamics:

$$I\ddot{\theta}(t) = T_c - mglsin(\theta(t))$$

State-space model with $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\frac{g}{l}sin(x_1) + \frac{1}{ml^2}u$$
$$y = x_1$$

Generalization

Example 1

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, u) = x_2 \\
\dot{x}_2 &= f_2(x_1, x_2, u) = g - \frac{cu^2}{mx_1^2} = 10 - \left(\frac{u}{3.87x_1}\right)^2 \\
y &= h(x_1, x_2, u) = x_1
\end{aligned}$$

Example 2

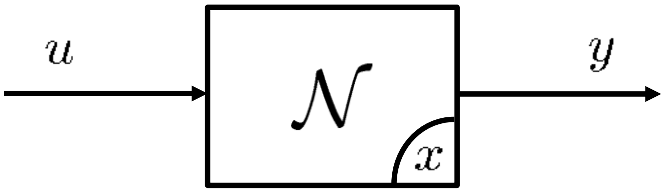
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} sin(x_1) + \frac{1}{ml^2} u$$

$$y = x_1$$

General form:

$$\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{aligned}$$



Nonlinear system and linearization

Nonlinear system $\mathcal{N}$:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t))\end{aligned}$$

Equilibrium point (state/input pair)

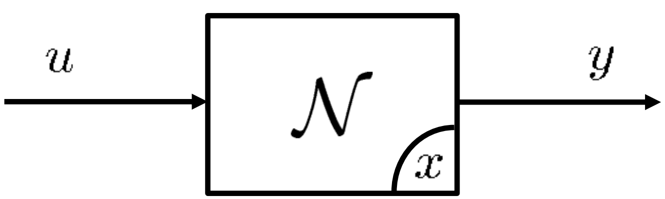
$$\bar{x} \in \mathbf{R}^n, \bar{u} \in \mathbf{R}^m$$

satisfying

$$f(\bar{x}, \bar{u}) = 0_n$$

Associated equilibrium output

$$\bar{y} := h(\bar{x}, \bar{u}) \in \mathbf{R}^q$$



Behavior at equilibrium point:

if
$$\begin{aligned}x(0) &= \bar{x} \\ u(t) &= \bar{u}, \quad \forall t \geq 0\end{aligned}$$

then
$$\begin{aligned}x(t) &= \bar{x} \quad \forall t \geq 0 \\ y(t) &= \bar{y} \quad \forall t \geq 0\end{aligned}$$

Behavior *near* equilibrium point

- Express in new variables

$$\begin{aligned}\eta(t) &:= x(t) - \bar{x} \\ v(t) &:= u(t) - \bar{u} \\ w(t) &:= y(t) - \bar{y}\end{aligned}$$

  which just represent *offset* from equilibrium point

- How are these deviation variables related?
- "near" refers to (η, v) being small

Nonlinear system and linearization

Express Nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

in new coordinates (*deviations*)

$$\begin{aligned} \eta(t) &:= x(t) - \bar{x} \\ v(t) &:= u(t) - \bar{u} \\ w(t) &:= y(t) - \bar{y} \end{aligned}$$

Exact Dynamics are

$$\begin{aligned} \dot{\eta}(t) &= \dot{x}(t) \\ &= f(\bar{x} + \eta(t), \bar{u} + v(t)) \end{aligned}$$

Conclusion: while the deviations from equilibrium,

$$(u(t) - \bar{u}, x(t) - \bar{x})$$

both remain small, the deviations from equilibrium are *approximately* governed by the linear ODE

$$\dot{\eta}(t) = A\eta(t) + Bv(t)$$

What if $(\eta(t), v(t))$ are "small"? Then the right-hand side is ≈

$$f(\bar{x} + \eta(t), \bar{u} + v(t)) \approx f(\bar{x}, \bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} \eta(t) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} v(t)$$

$$f(\bar{x}, \bar{u}) = 0_n \qquad := A \in \mathbf{R}^{n \times n} \qquad := B \in \mathbf{R}^{n \times m}$$

Nonlinear system and linearization

Express Nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

in new coordinates (*deviations*)

$$\begin{aligned} \eta(t) &:= x(t) - \bar{x} \\ v(t) &:= u(t) - \bar{u} \\ w(t) &:= y(t) - \bar{y} \end{aligned}$$

Exact output is

$$y(t) = h(\bar{x} + \eta(t), \bar{u} + v(t))$$

<u>Conclusion</u>: while the (deviations from equilibrium)

$$(u(t) - \bar{u}, x(t) - \bar{x})$$

both remain small, the deviations from equilibrium are approximately governed by the linear state-space model

$$\begin{aligned} \dot{\eta}(t) &= A\eta(t) + Bv(t) \\ y(t) - \bar{y} =: \quad w(t) &= C\eta(t) + Dv(t) \end{aligned}$$

What if $(\eta(t), v(t))$ are "small"? Then the right-hand side is ≈

$$h(\bar{x} + \eta(t), \bar{u} + v(t)) \approx h(\bar{x}, \bar{u}) + \left.\frac{\partial h}{\partial x}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} \eta(t) + \left.\frac{\partial h}{\partial u}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} v(t)$$

$h(\bar{x}, \bar{u}) \to \bar{y}$

$\left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}, u=\bar{u}} \to\ := C \in \mathbf{R}^{q \times n}$

$\left.\frac{\partial h}{\partial u}\right|_{x=\bar{x}, u=\bar{u}} \to\ := D \in \mathbf{R}^{q \times m}$

Nonlinear system and linearization

Nonlinear system $\mathcal{N}$:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

Equilibrium point (state/input pair)

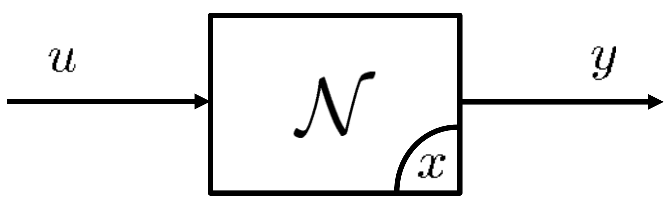
$$\bar{x} \in \mathbf{R}^n, \bar{u} \in \mathbf{R}^m$$

satisfying

$$f(\bar{x}, \bar{u}) = 0_n$$

Associated equilibrium output

$$\bar{y} := h(\bar{x}, \bar{u}) \in \mathbf{R}^q$$



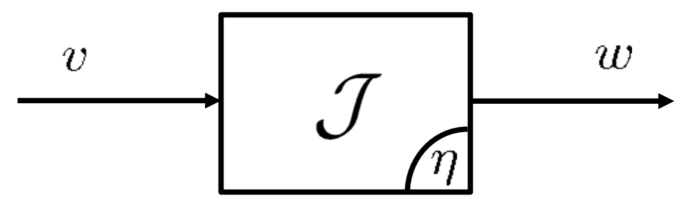
Jacobian Linearization:

$$A := \left.\frac{\partial f}{\partial x}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} \qquad B := \left.\frac{\partial f}{\partial u}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}}$$

$$C := \left.\frac{\partial h}{\partial x}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}} \qquad D := \left.\frac{\partial h}{\partial u}\right|_{\substack{x = \bar{x} \\ u = \bar{u}}}$$

Linear system $\mathcal{J}$:

$$\begin{aligned} \dot{\eta}(t) &= A\eta(t) + Bv(t) \\ w(t) &= C\eta(t) + Dv(t) \end{aligned}$$

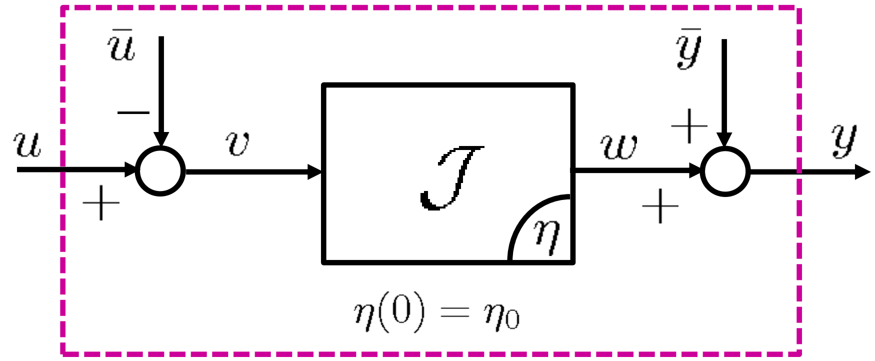


What is the relationship between these?

Nonlinear system and linearization

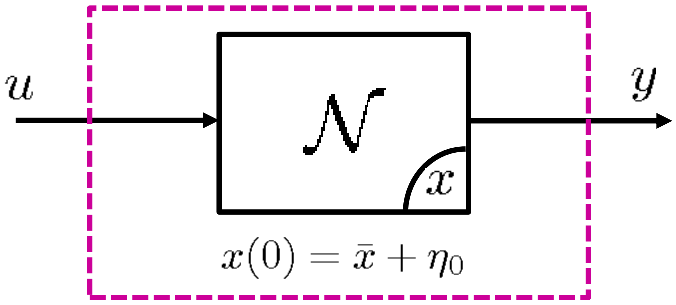
The shifted linear system (input u, state η, output y)

$$\dot{\eta}(t) = A\eta(t) + B\left(u(t) - \bar{u}\right)$$
$$y(t) = C\eta(t) + D\left(u(t) - \bar{u}\right) + \bar{y}$$
$$\eta(0) = \eta_0$$



behaves "approximately" like the appropriately initialized nonlinear system

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$
$$x(0) = \bar{x} + \eta_0$$



as long as the variables

$$\left(u(t) - \bar{u}, x(t) - \bar{x}\right)$$

remain "small"

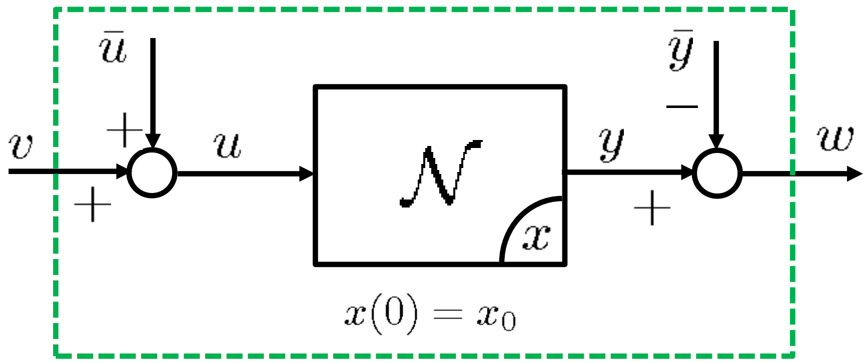
Nonlinear system and linearization (alternative view)

While the variables

$$(v(t), x(t) - \bar{x})$$

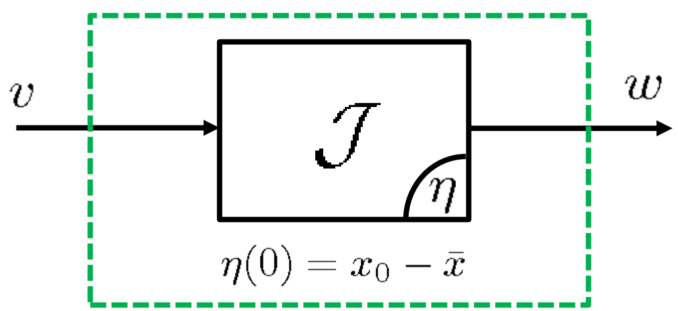
remain small, the shifted nonlinear system (input *v*, state *x*, output *w*)

$$\dot{x}(t) = f(x(t), \bar{u} + v(t))$$
$$w(t) = h(x(t), \bar{u} + v(t)) - \bar{y}$$
$$x(0) = x_0$$



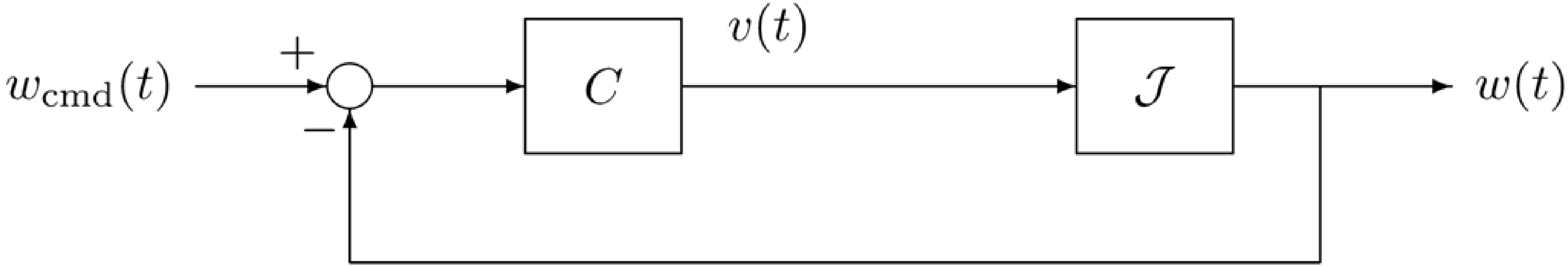
behaves "approximately" like the appropriately initialized linear system

$$\dot{\eta}(t) = A\eta(t) + Bv(t)$$
$$w(t) = C\eta(t) + Dv(t)$$
$$\eta(0) = x_0 - \bar{x}$$



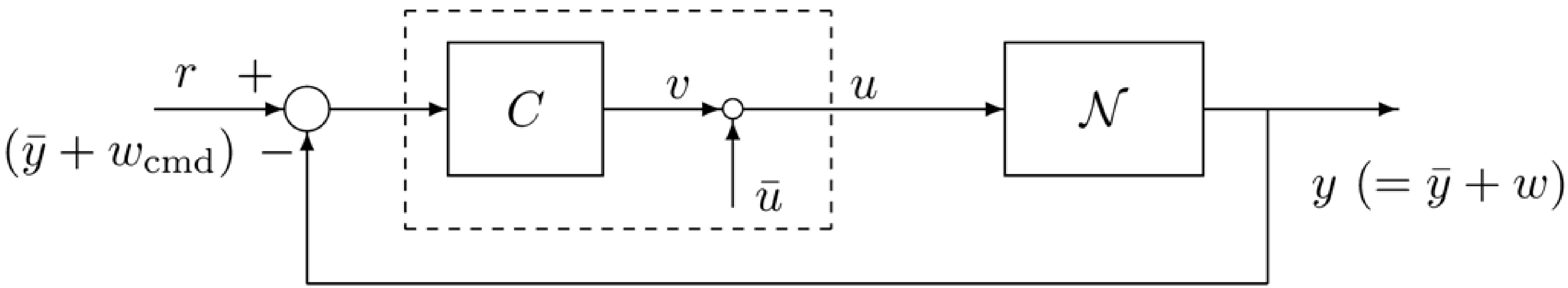
Use in feedback design

Design controller to regulate the linearized system



Here w_{cmd} represents desired value of w, in other words,

$$(y(t) - \bar{y})_{\mathrm{desired}} = w_{\mathrm{cmd}}(t) \quad \Longrightarrow \quad y_{\mathrm{desired}}(t) = \bar{y} + w_{\mathrm{cmd}}(t)$$



Background: Derivative

Suppose *f* is a real-valued function of a single real variable, notated

$$f : \mathbf{R} \to \mathbf{R}$$

The function is *differentiable at x* if the limit below exists

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

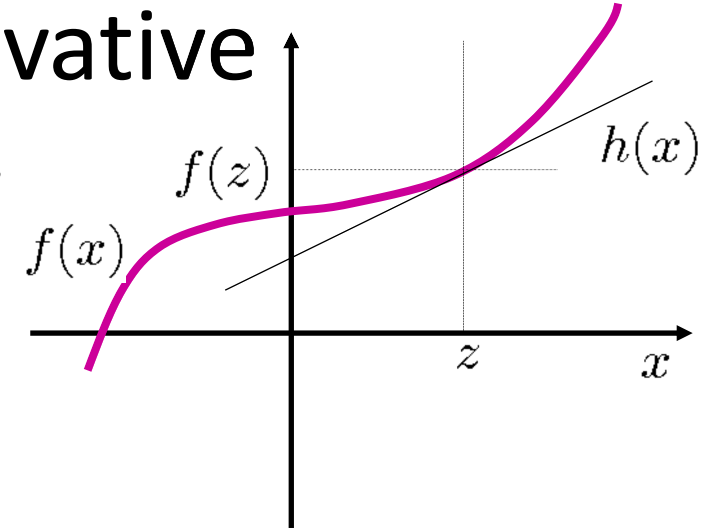
If so, then the *derivative of f, at x* is notated and defined as

$$f'(x) := \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

Interpretation: slope of tangent line to graph of f, at the point

$$(x, f(x))$$

Background: Derivative



Pick fixed value *z*, define function $h : \mathbf{R} \rightarrow \mathbf{R}$

$$h(x) := f(z) + f'(z)(x - z) \quad \forall x \in \mathbf{R}$$

Facts about *h*:

- Linear function
  - graph is a straight line
- The slope of *h* is constant, and $h'(x) = f'(z)$ for all $x \in \mathbf{R}$
- $h(z) = f(z)$

The function *h* is the only linear function with these two properties

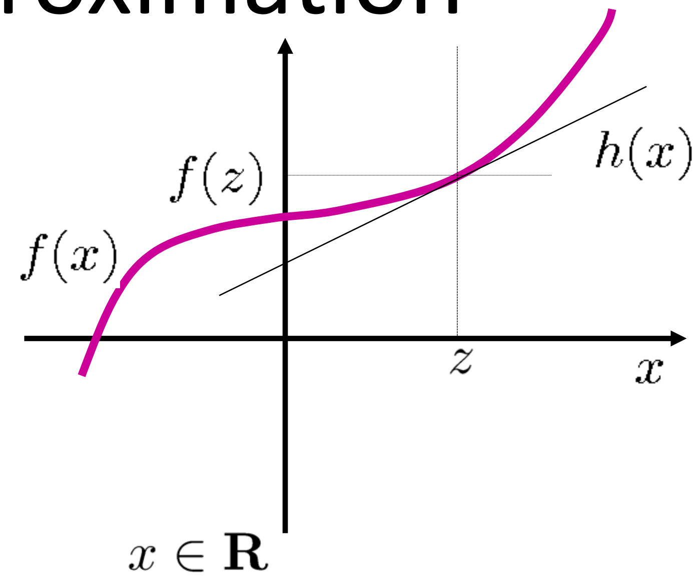
- $h(x) = f(x)$ at $x = z$
- $h'(x) = f'(x)$ at $x = z$

The function *h* is called the *linear approximation to f at x=z*

Accuracy of linear approximation

The difference between f and h, near x=z is small

$$h(x) := f(z) + f'(z)(x - z) \quad \forall x \in \mathbf{R}$$



Hence $f(x) - h(x) = f(x) - f(z) - f'(z)(x - z)$

and clearly $\lim_{x \to z} f(x) - h(x) = 0$

More is true, consider

$$\lim_{x \to z} \frac{f(x) - h(x)}{x - z} = \lim_{x \to z} \frac{f(x) - f(z) - f'(z)(x - z)}{x - z}$$

$$= \lim_{x \to z} \frac{f(x) - f(z)}{x - z} - \frac{f'(z)(x - z)}{x - z}$$

$$= \lim_{x \to z} \frac{f(x) - f(z)}{x - z} - \lim_{x \to z} \frac{f'(z)(x - z)}{x - z}$$

$$= 0$$

Multivariariate Partial Derivative

Let f be a function of x_1 and x_2. Pick fixed value $z=[z_1,z_2]^T$, define $h : \mathbf{R}^2 \to \mathbf{R}$

$$h(x) := f(z) + \left.\frac{\partial f}{\partial x_1}\right|_{x=z} (x_1 - z_1) + \left.\frac{\partial f}{\partial x_2}\right|_{x=z} (x_2 - z_2) \quad \forall x \in \mathbf{R^2}$$

$$= f(z) + \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \right]_{x=z} \left[\begin{array}{c} x_1 - z_1 \\ x_2 - z_2 \end{array} \right] \quad \forall x \in \mathbf{R^2}$$

$$= f(z) + \nabla f(x)|_{x=z}^{T} (x - z) \quad \forall x \in \mathbf{R^2}$$

Called the gradient of $f(x)$

Facts about h:

- Linear function
  - graph is a plane
- The partial derivative of h is constant, and $h'(x) = f'(z)$
- $h(z) = f(z)$

The function h is the <u>only</u> linear function with these two properties

- $h(x) = f(x) \text{ at } x = z$
- $\nabla h(x) = \nabla f(x) \text{ at } x = z$

The function h is called the *linear approximation to f at x=z*

Example: Tank System

Hot and Cold supplies (fixed temperatures)

$$T_C, T_H$$

Hot and cold inflows (eg., m^3/sec)

$$q_C, q_H$$

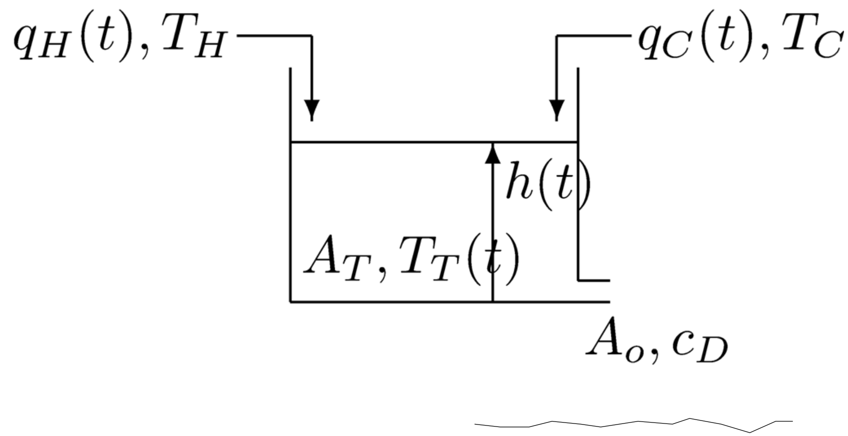
Perfect instantaneous mixing

– Temperature in Tank (assumed uniform)

$$T_T$$

Orifice outflow: Area, discharge coefficient

$$A_o, c_D$$



$$c_D A_o \sqrt{2gh(t)}$$

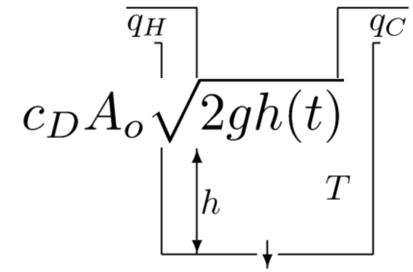
Mass Balance

$$
\begin{aligned}
\dot{h}(t) &= \frac{1}{A_T}\left(q_C(t) + q_H(t) - c_D A_o \sqrt{2gh(t)}\right) \\
\dot{T}_T(t) &= \frac{1}{h(t)A_T}\left(q_C(t)\left[T_C - T_T(t)\right] + q_H(t)\left[T_H - T_T(t)\right]\right)
\end{aligned}
$$

Nonlinear system and linearization

Define state and input vectors as

$$x(t) := \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} \quad , \quad u(t) := \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}$$



Then, with $f_1 : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$

$$\begin{aligned} \dot{h}(t) &= \frac{1}{A_T}\left(q_C(t) + q_H(t) - c_D A_o \sqrt{2gh(t)}\right) \\ \dot{T}_T(t) &= \frac{1}{h(t)A_T}\left(q_C(t)\left[T_C - T_T(t)\right] + q_H(t)\left[T_H - T_T(t)\right]\right) \end{aligned}$$

And

$$f_2(x_1, x_2, u_1, u_2) = \frac{1}{x_1 A_T}\left(u_1\left[T_C - x_2\right] + u_2\left[T_H - x_2\right]\right)$$

$$f_1(x_1, x_2, u_1, u_2) = \frac{1}{A_T}\left(u_1 + u_2 - c_D A_o \sqrt{2gx_1}\right)$$

Dynamic equations are of the form

$$\begin{aligned} \dot{x}_1(t) &= f_1(x(t), u(t)) \\ \dot{x}_2(t) &= f_2(x(t), u(t)) \end{aligned}$$

Equilibrium Points

Equilibrium points are characterized by $f(\bar{x}, \bar{u}) = 0$. In this case, with

$$f_1(x_1, x_2, u_1, u_2) = \frac{1}{A_T}\left(u_1 + u_2 - c_D A_o \sqrt{2 g x_1}\right)$$
$$f_2(x_1, x_2, u_1, u_2) = \frac{1}{x_1 A_T}\left(u_1 \left[T_C - x_2\right] + u_2 \left[T_H - x_2\right]\right)$$

Writing with barred-quantities, and setting to 0 gives (assuming $\bar{x}_1 \neq 0$)

$$\begin{bmatrix} 1 & 1 \\ T_C - \bar{x}_2 & T_H - \bar{x}_2 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} c_D A_o \sqrt{2 g \bar{x}_1} \\ 0 \end{bmatrix}$$

$\text{invertible} \Leftrightarrow T_C \neq T_H$

For any choice of $\bar{x}$, there is a unique equilibrium input $\bar{u}$, given by

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \frac{1}{T_H - T_C} \begin{bmatrix} T_H - \bar{x}_2 & -1 \\ \bar{x}_2 - T_C & 1 \end{bmatrix} \begin{bmatrix} c_D A_o \sqrt{2 g \bar{x}_1} \\ 0 \end{bmatrix}$$

Equilibrium Points

For any choice of $\bar{x}$, there is a unique equilibrium input $\bar{u}$, given by

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \frac{1}{T_H - T_C} \begin{bmatrix} T_H - \bar{x}_2 & -1 \\ \bar{x}_2 - T_C & 1 \end{bmatrix} \begin{bmatrix} c_D A_o \sqrt{2g\bar{x}_1} \\ 0 \end{bmatrix}$$

invertible $\Leftrightarrow T_C \neq T_H$

which gives

$$\bar{u}_1 = \frac{c_D A_o \sqrt{2g\bar{x}_1}\,(T_H - \bar{x}_2)}{T_H - T_C} \quad , \quad \bar{u}_2 = \frac{c_D A_o \sqrt{2g\bar{x}_1}\,(\bar{x}_2 - T_C)}{T_H - T_C}$$

The inputs u_i represent flow into tank, before irreversible mixing, and should be restricted to be nonnegative

Since the u_i represent flow rates **into** the tank, physical considerations restrict them to be nonegative real numbers. This implies that $\bar{x}_1 \geq 0$ and $T_C \leq \bar{T}_T \leq T_H$. Looking at the differential equation for T_T, we see that its rate of change is inversely related to h. Hence, the differential equation model is valid while $h(t) > 0$, so we further restrict $x_1 > 0$. Under those restrictions, the state $\bar{x}$ is indeed an equilibrium point, and there is a unique equilibrium input given by the equations above.

Partial derivatives

Obtain the “A” and “B” matrices, by first taking partial derivatives

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{g c_D A_o}{A_T \sqrt{2 g x_1}} & 0 \\ -\frac{u_1[T_C - x_2] + u_2[T_H - x_2]}{x_1^2 A_T} & \frac{-(u_1 + u_2)}{x_1 A_T} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{A_T} & \frac{1}{A_T} \\ \frac{T_C - x_2}{x_1 A_T} & \frac{T_H - x_2}{x_1 A_T} \end{bmatrix}$$

Pick some values for the constants

$$T_C = 10^\circ, T_H = 90^\circ, A_T = 3\mathtt{m}^2, A_o = 0.05\mathtt{m}, c_D = 0.7$$

Compute linearization at 4 different equilibrium points

$$(\bar{h} = 1\mathtt{m}, \bar{T}_T = 25^\circ) \quad (\bar{h} = 3\mathtt{m}, \bar{T}_T = 25^\circ)$$

$$(\bar{h} = 1\mathtt{m}, \bar{T}_T = 75^\circ) \quad (\bar{h} = 3\mathtt{m}, \bar{T}_T = 75^\circ)$$

Partial derivatives

Obtain the "A" and "B" matrices, by first taking partial derivatives

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{g c_D A_o}{A_T\sqrt{2gx_1}} & 0 \\ -\frac{u_1[T_C-x_2]+u_2[T_H-x_2]}{x_1^2 A_T} & \frac{-(u_1+u_2)}{x_1 A_T} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{A_T} & \frac{1}{A_T} \\ \frac{T_C-x_2}{x_1 A_T} & \frac{T_H-x_2}{x_1 A_T} \end{bmatrix}$$

Results for:

$(\bar{h}, \bar{T}_T) = (1\mathtt{m}, 25°)$

$$\bar{u}_1 = \bar{q}_C = 0.126 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.029$$

$$A = \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.517 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -5.00 & 21.67 \end{bmatrix}$$