Essentials of Control Systems

Discretization of Continuous-time Transfer-function Systems

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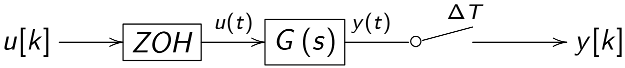
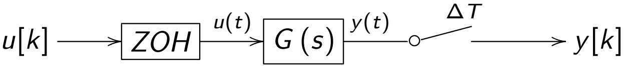
Overview

- Consider the discrete-time controller implementation scheme



  where $u[k]$ and $y[k]$ have the same sampling time.
- for this note, we use $[k]$ to distinguish DT signals from their CT counter parts
- Goal: to derive the transfer function from $u[k]$ to $y[k]$.
- Solution concept: let $u[k]$ be a discrete-time impulse (whose Z transform is 1) and obtain the Z transform of $y[k]$.

Solution



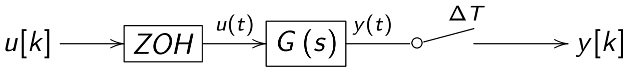
- $u[k]$ is a DT impulse $\Rightarrow$ after ZOH

$$u(t) = \begin{cases} 1, & 0 \leq t < \Delta T \\ 0, & \text{otherwise} \end{cases} = 1(t) - 1(t - \Delta T) \Longrightarrow U(s) = \frac{1 - e^{-s\Delta T}}{s}$$

- Hence

$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1 - e^{-s\Delta T}}{s}\right] = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] - \mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]$$

Solution



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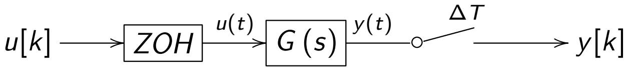
- Sampling $y(t)$ at ΔT and performing Z transform give:

$$G(z) = \mathcal{Z}\left\{ \underbrace{\overbrace{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]}^{\tilde{y}(t)}\Bigg|_{t=k\Delta T}}_{\triangleq \tilde{y}[k]} - \underbrace{\overbrace{\mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]}^{\tilde{y}(t-\Delta T)}\Bigg|_{t=k\Delta T}}_{=\tilde{y}[k-1]!!!} \right\}$$

$$= \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Bigg|_{t=k\Delta T} \right\} - z^{-1}\mathcal{Z}\left\{ \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Bigg|_{t=k\Delta T} \right\}$$

Solution



Fact

The zero order hold equivalent of $G(s)$ is

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{ \mathcal{L}^{-1}\left[G(s)\frac{1}{s} \right] \Big|_{t=k\Delta T} \right\}$$

where ΔT is the sampling time.

Example

Obtain the ZOH equivalent of

$$G(s) = \frac{a}{s+a}$$

Following the discretization procedures we have $\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$ and hence

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(t) - e^{-at}1(t)$$

Sampling at ΔT gives $1[k] - e^{-ak\Delta T}1[k]$, whose Z transform is

$$\frac{z}{z-1} - \frac{z}{z-e^{-a\Delta T}} = \frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})}$$

Hence the ZOH equivalent is

$$(1-z^{-1})\frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})} = \frac{1-e^{-a\Delta T}}{z-e^{-a\Delta T}}$$

Matlab command

In MATLAB, the function *c2d.m* computes the ZOH equivalent of a continuous-time transfer function, as well as other discrete equivalents. For

$$G(s) = \frac{1}{s^2}$$

and $\Delta T = 1$, the following scripts

```
T=1;
numG=1; denG=[1 0 0];
G = tf(numG,denG);
Gd = c2d(G,T);
```

produces the correct ZOH equivalent.

Exercise

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Find the zero order hold equivalent of $G(s) = e^{-Ls}$, $2\Delta T < L < 3\Delta T$, where ΔT is the sampling time.