Lecture Notes
Partial Differential Equations

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Contents

1 Basic concepts of PDEs

- A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of a function (call it u) that depends on two or more variables, often time t and one or several variables in space.
- The order of the highest derivative is called the **order** of the PDE.
- A PDE is linear if it is of the first degree in the unknown function u and its partial derivatives.
  - e.g. $\partial u/\partial t = c^2\partial^2 u/\partial x^2$ is a linear PDE
  - $(\partial u/\partial t)^2 = c^2\partial^2 u/\partial x^2$ is a nonlinear PDE

Important second-order PDEs

- one-dimensional wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$
- one-dimensional heat equation
$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \tag{2}$$
- two-dimensional Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$
- two-dimensional Poisson equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{4}$$
- two-dimensional wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{5}$$
- three-dimensional Laplace equation
$$\nabla^2 u \triangleq \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{6}$$

  where c is a positive constant, t is time, and x, y, z are Cartesian coordinates.

We often write u_x to denote $\partial u/\partial x$, u_{xx} to denote $\partial^2 u/\partial x^2$, etc. So, the two dimensional Laplace equation (3) can be equivalently written as

$$u_{xx} + u_{yy} = 0$$

One PDE can have many solutions. For instance

$$u = x^2 - y^2,\ u = e^x \cos y,\ u = \sin x \cosh y,\ u = \ln\left(x^2 + y^2\right)$$

are all solutions of the two-dimensional Laplace equation (3).

Usually a PDE is defined in some bounded domain D, giving some **boundary conditions** and/or **initial conditions**. These additional conditions are very important to define a unique solution for the PDE.

Theorem 1 (Fundamental theorem on superposition)**.** *If u_1 and u_2 are solutions of a homogeneous linear PDE in some region R, then*

$$u = c_1 u_1 + c_2 u_2$$

with any constant c_1 and c_2 is also a solution of the PDE in R.

2 PDEs solvable as ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form).

Example 2. Solve $u_{xx} - u = 0$.

This can be solved like $\ddot{u} - u = 0$, which has a solution $u = Ae^{-x} + Be^x$. The only difference is that A and B here may be functions of y. So the answer is

$$u\left(x, y\right) = A\left(y\right) e^x + B\left(y\right) e^{-x}$$

where $A\left(y\right)$ and $B\left(y\right)$ are arbitrary functions of y.

□

Example 3. Solve $u_{xy} = -u_x$.

Let $u_x = p$. Then $p_y = -p \Rightarrow \ln|p| = -y + c\left(x\right)$. Note again that $c\left(x\right)$ is now a function of x instead of a constant. So $p = c\left(x\right) e^{-y}$, i.e.

$$\frac{\partial u}{\partial x} = c\left(x\right) e^{-y}$$

Integration with respect to x gives

$$u\left(x, y\right) = f\left(x\right) e^{-y} + g\left(y\right),\ f\left(x\right) = \int c\left(x\right) dx$$

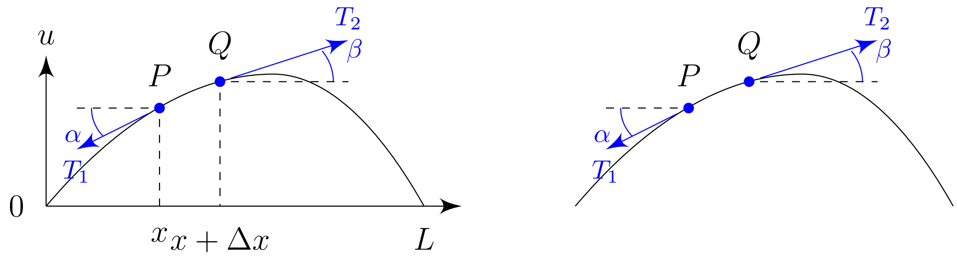


Figure 1: Vibration string

3 Vibrating string and wave equation

We derive the PDE modeling small transverse vibration of an elastic string, such as a violin string. Consider the illustrative picture above. The string is placed along the x-axis, stretched to length L, and fastened at the ends $x = 0$ and $x = L$. The string is distorted at $t = 0$, and released to vibrate. The problem is to determine the string deflection $u(x,t)$ at a point $x \in [0, L]$.

Assumptions:

- The tension caused by stretching the string is so large that the action of the gravitation force can be neglected;
- The deflection happens in the vertical plane. Every particle of the string moves strictly vertically. The deflection and the slope at every point of the string always remain small in absolute value.

3.1 PDE modeling

Consider the forces acting on a small portion of the string. This method is typical of modeling in mechanics and many other Engineering applications.

Recall Fig. 1. There is no acceleration in the x direction. Hence the horizontal components of the tension must be constant, i.e.

$$T_1 \cos\alpha = T_2 \cos\beta = T \tag{7}$$

By Newton's second law, in the vertical direction, we have

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

where ρ is the mass of the undeflected string per unit length.

Dividing the last equation by (7) yields

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \tag{8}$$

Notice that

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x}$$

and

$$\tan\beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}$$

Hence (8) is equivalent to

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Taking the limit case of $\Delta x \to 0$, we get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ c^2 = \frac{T}{\rho}}$$

which is the **one-dimensional wave equation.**

3.2 Solution by separating variables and Fourier series

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{9}$$

with two boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0$$

and two initial conditions (position and velocity)

$$u(x,0) = f(x),\ u_t(x,0) = g(x),\ 0 \le x \le L$$

It turns out that for PDEs in the structure of (9), a common method called separating variables can be applied.

Solution steps:

1. **"method of separating variables":** set

$$\boxed{u(x,t) = F(x)\,G(t)} \tag{10}$$

   to obtain two ODEs: one for $F(x)$ and one for $G(t)$[1]

2. solve the two individual ODEs

3. use Fourier series to compose the final solution

[1] Similar idea applies to general multi-variable functions.

Details:

Step 1: substituting (10) into (9) gives

$$F(x)\frac{d^2G(t)}{dt^2} = c^2G(t)\frac{d^2F(x)}{dx^2}$$

namely

$$\frac{\ddot{G}}{c^2G} = \frac{F''}{F}$$

The left side is a function of t only; and the right side is a function of x only. Hence it must be that

$$\boxed{\frac{\ddot{G}}{c^2G} = \frac{F''}{F} = k}$$

Therefore we have two ODEs:

$$F'' - kF = 0 \tag{11}$$

and

$$\ddot{G} - c^2kG = 0 \tag{12}$$

Step 2: satisfying the boundary conditions

Step 2.1: (11) has the boundary condition

$$u(0,t) = F(0)G(t) = 0$$
$$u(L,t) = F(L)G(t) = 0$$

The case for $G(t) \equiv 0$ is not practically interesting. Hence we need

$$F(0) = F(L) = 0$$

It turns out that k should be negative. Otherwise, if $k = 0$, then $F(x) = ax + b$ is the solution of (11), and the boundary condition gives $a = b = 0$; if k is positive, say $k = \mu^2$, then we have $F(x) = Ae^{\mu x} + Be^{-\mu x}$, and the boundary condition again gives $F \equiv 0$. Thus, we can let $k = -p^2$. Then the ODE becomes

$$F'' + p^2F = 0$$

whose solution is

$$F(x) = A\cos px + B\sin px$$

Adding the boundary condition in this case gives

$$F(0) = A = 0,\ F(L) = B\sin pL = 0$$

Hence for a practically meaningful solution, it must be that $\sin pL = 0$. Thus

$$pL = n\pi \Rightarrow p = \frac{n\pi}{L},\ n \text{ is an integer}$$

and

$$\boxed{F(x) = F_n(x) = B \sin \frac{n\pi}{L} x}$$

Step 2.2: with the above discussions, we know that

$$k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$$

So (12) becomes

$$\ddot{G} + \lambda_n^2 G = 0, \;\; \lambda_n = cp = \frac{cn\pi}{L}$$

whose solution is

$$\boxed{G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t}$$

Hence based on (10) we have

$$u_n(x,t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) B \sin \frac{n\pi}{L} x$$

The scalar B is redundant. We can absorb it into B_n and B_n^*, to get

$$\boxed{u_n(x,t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x}$$

which is called the eigenfunction or characteristic function of the PDE; and λ_n's are called the eigenvalues of the vibrating string.

Step 3: The eigenfunctions satisfy the PDE and the boundary equation. However a single u_n generally does not satisfy the initial conditions. This is addressed by noting that the PDE is linear and homogeneous, hence a linear combination of the eigenfunctions also is a solution. Let

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x}$$

we can enforce the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x) \tag{13}$$

We can extend $f(x)$ to $x < 0$ and $x > L$ so that it is an odd periodic function with period $2L$ (what we are interested is only the region where $x \in [0, L]$). So (13) is in the form of a Fourier series. We thus need B_n to be the Fourier series coefficients:

$$\boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx} \tag{14}$$

There is yet another coefficient B_n^* to be determined. The second initial condition is

$$u_t(x,0) = g(x)$$

in other words [after using (14)],

$$\sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x = g(x)$$

Hence

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$$

Using $\lambda_n = cn\pi/L$, we get

$$\boxed{B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx}$$

3.3 Alternative representations

It turns out the solution derived in the past subsection can have simplified representations.

For simplicity, we consider $g(x) = 0$ in this subsection. Then $B_n^* = 0$.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x, \ \ \lambda_n = \frac{cn\pi}{L}$$

in other words

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L} ct\right) \sin\left(\frac{n\pi}{L} x\right)$$

Alternatively, we can write it as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \frac{1}{2} \left\{ \sin\left[\frac{n\pi}{L}(x - ct)\right] + \sin\left[\frac{n\pi}{L}(x + ct)\right] \right\} \tag{15}$$

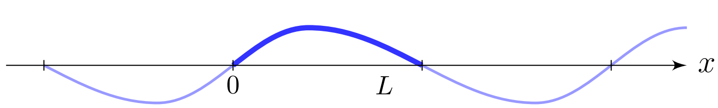
Recall from the initial condition (13), that

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)$$

We see that (15) is nothing but

$$u(x,t) = \frac{1}{2}\left[f^*(x - ct) + f^*(x + ct)\right] \tag{16}$$

where f^* is the odd periodic extension of f, as shown below



4 Heat flow

The heat flow problem is concerned with the temperature u in a body in space. The physical model obeys the heat equation

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 a = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

where $c^2 = K / (\rho\sigma)$; K is the thermal conductivity constant; σ is the specific heat; ρ is the density of the material of the body.

Intuitively, the PDE describes the energy conservation in the body and its environment. The left hand side is related to the temperature change w.r.t. time; the right hand side is related to the heat flow exchange in the body.

The PDE has many other applications. For instance, it also models chemical diffusion processes of one substance or gas into another.

4.1 One-dimensional heat equation

Consider a long thin bar, with constant cross section and homogeneous material, within which heat flows in the x-direction only. The heat equation becomes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},\ c^2 = \frac{K}{\rho\sigma} \tag{17}$$

Although the equation looks very similar to the wave equation, the solutions will be shown to be quite different here.

Consider the boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0,\ \forall t \geq 0$$

and the initial condition

$$u(x,0) = f(x)$$

where it is assumed that $f(x)$ is piecewise continuous on $[0, L]$ and has one-sided derivatives at all interior points of that interval.

The same solution technique applies here–first separate $u(x,t)$ as $F(x)G(t)$, then solve two separate ODEs, and finally use Fourier series to synthesize.

- **Step 1:** two ODEs from the PDE.
  - Let $u(x,t) = F(x)G(t)$. Then (17) gives

$$F\dot{G} = c^2 F'' G$$

Dividing by the nonzero c^2FG ($c^2FG = 0$ is not practically interesting) gives

$$\frac{\dot{G}}{c^2G} = \frac{F''}{F}$$

The left side depends only on t and the right side only on x. Hence both sides must equal a constant k.

– Similar to the wave equation, you can show that it is only interesting to have a negative k. Let $k = -p^2$. We have

$$F'' + p^2F = 0 \tag{18}$$
$$\dot{G} + c^2p^2G = 0 \tag{19}$$

- **Step 2:** solve (18) and (19) with the boundary condition constraints. For (18), the general solution is

$$F(x) = A\cos px + B\sin px$$

To satisfy the boundary condition, it must be that

$$u(0,t) = F(0)G(t) = 0$$

and

$$u(L,t) = F(L)G(t) = 0$$

It is not interesting to have $G(t) \equiv 0$. So $F(0) = 0$ and $F(L) = 0$, yielding

$$A = 0$$

and

$$\sin(pL) = 0 \Rightarrow p = \frac{n\pi}{L},\ n = 1, 2, \ldots$$

We finally get

$$F(x) = B\sin\frac{n\pi x}{L},\ n = 1, 2, \ldots \tag{20}$$

With (19) and $p = n\pi/L$, we have

$$\dot{G} + \lambda_n^2 G = 0,\ \lambda_n = \frac{cn\pi}{L}$$

whose solution is

$$G_n(t) = B_n e^{-\lambda_n^2 t} \tag{21}$$

The two constant scalars in (20) and (21) can be combined. Taking $B = 1$, we get

$$u_n(x,t) = F_n(t)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \tag{22}$$

- **Step 3:** use Fourier series to solve the entire problem. To additionally satisfy the initial conditions, the eigenfunctions (22) are combined to give

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-\lambda_n^2 t},\ \lambda_n=\frac{cn\pi}{L}} \tag{23}$$

  Adding the constraint

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}=f(x)$$

  we see that B_n has to be the coefficients of the Fourier series, i.e.

$$\boxed{B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

Observations:

Note the exponential factor $e^{-\lambda_n^2 t}$ in the solution. The temperature of the bar will approach to zero as t approaches to infinity. The decay rate depends on the length L and material properties c of the bar.

Example 4. A copper bar with length $L=80cm$ has an initial temperature $100\sin(3\pi x/80)$ degrees C. The ends are kept at 0 degree C. How long will it take for the maximum temperature in the bar to drop to 50 degrees C? Copper has a density of $8.92g/cm^3$, a specific heat of $0.092cal/(g\cdot^o C)$, and a thermal conductivity of $0.95cal\,(cm\cdot sec\cdot^o C)$

Solution: We have

$$c^2=\frac{K}{\sigma\rho}=\frac{0.95}{0.092\cdot 8.92}=1.158\left[\mathrm{cm}^2/\mathrm{sec}\right]$$

From the initial condition and (23)

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}=f(x)=100\sin\frac{3\pi x}{80}$$

By inspection,

$$n=3,\ B_3=100,\ B_1=B_2=B_4=\cdots=0$$

Thus

$$u(x,t)=100\sin\frac{3\pi x}{L}e^{-\lambda_3^2 t},\ \lambda_3^2=\frac{3^2c^2\pi^2}{L^2}=0.01607$$

For the maximum temperature to drop 50 degrees C, we need

$$e^{-\lambda_3^2 t}=0.5\Rightarrow t=\frac{\ln 0.5}{-0.01607}=43\,[sec]$$

4.2 Steady two-dimensional heat problems

We show another application of the method of separating variables in two-dimensional steady (time-independent, i.e. $\partial u/\partial t = 0$) heat problem. The heat equation is

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

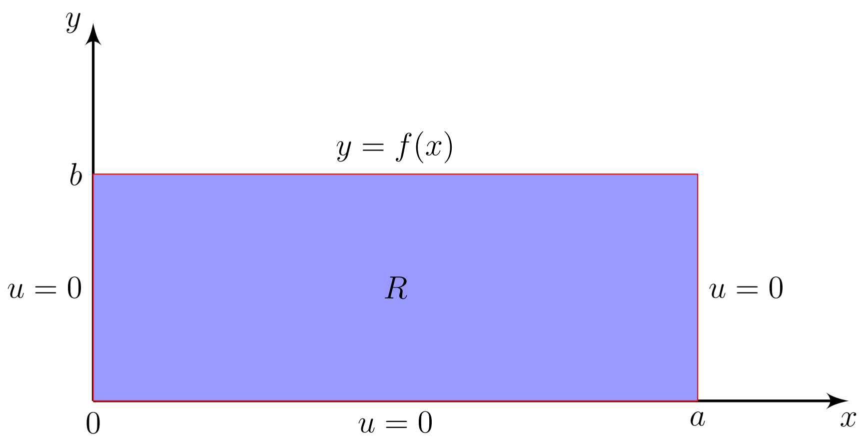
As $\partial u/\partial t = 0$, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Several boundary value problems (BVPs) can be considered in a region R of the xy plane and a given boundary condition on the boundary curve C of R:

- First BVP or Dirichlet Problem: if u is prescribed on C
- Second BVP or Neumann Problem: if the normal derivative $u_n = \partial u/\partial n$ is prescribed on C
- Third BVP, Mixed BVP, or Robin Problem: if u is prescribed on a portion of C and u_n on the rest of C

Dirichlet Problem in a rectangle R:



Consider the picture above. The boundary conditions are as specified. Applying the method of separating variables $u(x, y) = F(x) G(y)$ to

$$u_{xx} + u_{yy} = 0$$

we have

$$\frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{G} \frac{d^2 G}{dy^2} = -k$$

which gives

$$\frac{d^2 F}{dx^2} + kF = 0$$

and

$$\frac{d^2G}{dy^2} - kG = 0$$

The boundary conditions imply that

$$F(0) = 0, \;\; F(a) = 0$$

Similar as what we have done in the previous examples, it can be obtained that k must be positive:

$$k = \left(\frac{n\pi}{a}\right)^2$$

and the solution to the first ODE is

$$F(x) = F_n(x) = \sin\frac{n\pi}{a}x$$

For the second ODE

$$\frac{d^2G}{dy^2} - kG = \frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

the solution is

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

Applying the boundary condition on the y direction yields that $G(0) = 0$, namely $A_n = -B_n$. This gives

$$G_n(y) = A_n\left(e^{n\pi y/a} - e^{-n\pi y/a}\right) = 2A_n \sinh\frac{n\pi y}{a}$$

We can thus write $A_n^* = 2A_n$ and get

$$u_n(x,y) = F_n(x)\,F_n(y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Finally, we consider the infinite series

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

to get a solution that also satisfies the boundary condition $u(x,b) = f(x)$. It is required that

$$u(x,b) = \sum_{n=1}^{\infty}\left(A_n^*\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = f(x)$$

which shows that $A_n^*\sinh\frac{n\pi b}{a}$ must be the Fourier coefficients of $f(x)$:

$$A_n^*\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

Summarizing:

$$u(x,y) = \sum_{n=1}^{\infty} A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

where

$$A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

5 *Solving 2nd-order PDEs via the method of characteristics

This section discusses an alternative solution technique for second-order PDEs.

Big picture Certain PDEs are easy to solve. For example

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0 \tag{24}$$

can be readily solved by two successive integrations:

$$\frac{\partial u}{\partial v} = h(v)$$

and then

$$u = \int h(v)\,dv + \psi(w) = \phi(v) + \psi(w)$$

In terms of x and t, we thus have

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)} \tag{25}$$

It turns out that many PDEs can be transformed to the form of (24).

One-dimensional wave equation In

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{26}$$

introduce new variables

$$v = x + ct,\ \ w = x - ct$$

then u became a function of v and w.

By chain rule, we have

$$\begin{aligned}
u_x &= u_v v_x + u_w w_x = u_v + u_w \\
u_{xx} &= (u_v + u_w)_x = (u_v + u_w)_v\, v_x + (u_v + u_w)_w\, w_x \\
&= u_{vv} + u_{wv} + u_{vw} + u_{ww}
\end{aligned}$$

We assume that all the partial derivatives are continuous, so that $u_{wv} = u_{vw}$. Hence

$$u_{xx} = u_{vv} + 2u_{vw} + u_{ww} \tag{27}$$

Performing the above steps to the partial derivatives with respect to t, we get

$$\begin{aligned}
u_t &= u_v v_t + u_w w_t = c u_v - c u_w \\
u_{tt} &= c\,(u_v - u_w)_t = c\,(u_v - u_w)_v\, v_t - c\,(u_v + u_w)_w\, w_t \\
&= c^2\,(u_{vv} - 2u_{vw} + u_{ww})
\end{aligned} \tag{28}$$

Substituting (27) and (28) into (26) gives

$$c^2\left(u_{vv}-2u_{vw}+u_{ww}\right)=c^2\left(u_{vv}+2u_{vw}+u_{ww}\right)$$

in other words

$$u_{vw}=\frac{\partial^2 u}{\partial v\partial w}=0$$

which is in the form of (24)! Therefor the solution is in the form of (25).

We now add the boundary and initial conditions to obtain the detailed forms of ϕ and ψ in (25).

Differentiating (25) and applying the chain rule gives

$$u_t\left(x,t\right)=c\phi^{'}\left(x+ct\right)-c\psi^{'}\left(x-ct\right)$$

To satisfy the initial condition

$$u\left(x,0\right)=f\left(x\right),\ u_t\left(x,0\right)=g\left(x\right)$$

we must have

$$\begin{aligned}\phi\left(x\right)+\psi\left(x\right)&=f\left(x\right)\\ c\phi^{'}\left(x\right)-c\psi^{'}\left(x\right)&=g\left(x\right)\end{aligned}\tag{29}$$

Integrating the second equation yields

$$\phi\left(x\right)-\psi\left(x\right)=\phi\left(x_0\right)-\psi\left(x_0\right)+\int_{x_0}^{x}g\left(s\right)ds$$

which, combined with (29), gives

$$\begin{aligned}\phi\left(x\right)&=\frac{1}{2}f\left(x\right)+\frac{1}{2c}\int_{x_o}^{x}g\left(s\right)ds+\frac{1}{2}\left[\phi\left(x_0\right)-\psi\left(x_0\right)\right]\\ \psi\left(x\right)&=\frac{1}{2}f\left(x\right)-\frac{1}{2c}\int_{x_o}^{x}g\left(s\right)ds-\frac{1}{2}\left[\phi\left(x_0\right)-\psi\left(x_0\right)\right]\end{aligned}$$

Changing the notations of variables, we get

$$\begin{aligned}\phi\left(x+ct\right)&=\frac{1}{2}f\left(x+ct\right)+\frac{1}{2c}\int_{x_o}^{x+ct}g\left(s\right)ds+\frac{1}{2}\left[\phi\left(x_0\right)-\psi\left(x_0\right)\right]\\ \psi\left(x-ct\right)&=\frac{1}{2}f\left(x-ct\right)-\frac{1}{2c}\int_{x_o}^{x-ct}g\left(s\right)ds-\frac{1}{2}\left[\phi\left(x_0\right)-\psi\left(x_0\right)\right]\end{aligned}$$

After simplifications, we finally have

$$\boxed{u\left(x,t\right)=\frac{1}{2}\left[f\left(x+ct\right)+f\left(x-ct\right)\right]+\frac{1}{2c}\int_{x_o-ct}^{x+ct}g\left(s\right)ds}$$

which is an alternative representation of the solution from the Fourier series method.[2]

[2] Let $g\left(x\right)=0$. You can see that the solution is the same as that in (16).

General second-order PDEs solvable via the method of characteristics The method of characteristics generalizes the procedure discussed above. It concerns PDEs of the form

$$\boxed{Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)} \tag{30}$$

Example 5. For the 1-d wave equation, we have $y = ct$ and hence

$$u_{tt} - c^2 u_{xx} = c^2 (u_{yy} - u_{xx}) = 0$$

□

Example 6. For the 1-d heat equation, let $y = c^2 t$, then

$$u_t - c^2 u_{xx} = c^2 (u_y - u_{xx}) = 0$$

In general, (30) can be classified to three types:

Type	Defining condition	Example
hyperbolic	$AC - B^2 < 0$	wave equation
parabolic	$AC - B^2 = 0$	heat equation
elliptic	$AC - B^2 > 0$	Laplace equation

Similar as before, to solve the PDE, we introduce new variables v and w, which are functions of x and y. The choice of v and w obeys specific rules from engineering and mathematical experience. For the three types of PDEs, we have

Type	New variables	Normal form
hyperbolic	$v = \Phi$, $w = \Psi$	$u_{vw} = F_1$
parabolic	$v = x$, $w = \Phi = \Psi$	$u_{ww} = F_2$
elliptic	$v = \frac{1}{2}(\Phi + \Psi)$, $w = \frac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

where Φ and Ψ are from solving a characteristic ODE equation

$$A\left(y^{'}\right)^2 - 2By^{'} + C = 0$$

with $y^{'} = dy/dx$.

More specifically, solve the characteristic equation and write it in the form of $\Phi(x, y)$ =const and $\Psi(x, y)$ =const. For instance, if we have

$$u_{xx} + 4u_{yy} = 0$$

then $A = 1$, $B = 0$, and $C = 4$. This corresponds to the elliptic type of PDE. The characteristic equation is

$$\left(y^{'}\right)^2 + 4 = 0$$

yielding

$$y^{'} = \pm 2i,\ y = \int y^{'} dx + \text{const} = \pm 2ix + \text{const}$$

Hence

$$\Phi = y - 2ix,\ \Psi = y + 2ix$$

The new variables should be chosen as

$$v = \frac{1}{2}(\Phi + \Psi) = y$$
$$w = \frac{1}{2i}(\Phi - \Psi) = -2x$$

With the new variables, you can verify that the PDE is transformed into

$$u_{vv} + u_{ww} = 0$$

The derivation details are in "Methods of Mathematical Physics 2 vols" by Courant, R. and D. Hilbert.

Exercise 7. Transform the PDEs to normal forms and solve.

$$u_{xx} + 4u_{yy} = 0$$
$$u_{xx} - 2u_{xy} + u_{yy} = 0$$
$$u_{xx} - 6u_{xy} + 9u_{yy} = 0$$

6 Reference

[**EK**] Erwin Kreyszig, Advanced Engineering Mathematics, 10th edition