Lecture 6: Linear Quadratic Gaussian (LQG) Control

Big picture
LQ when there is Gaussian noise
LQG
Steady-state LQG

Big picture

in deterministic control design:

- state feedback: arbitrary pole placement for controllable systems
- observer provides (when system is observable) state estimation when not all states are available
- separation principle for observer state feedback control

we have now learned:

- LQ: optimal state feedback which minimizes a quadratic cost about the states
- KF: provides optimal state estimation

in stochastic control:

- the above two give the linear quadratic Gaussian (LQG) controller

Big picture

plant:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

assumptions:

- $w(k)$ and $v(k)$ are independent, zero mean, white Gaussian random processes, with

$$\mathrm{E}[w(k)w^T(k)] = W,\ \mathrm{E}[v(k)v^T(k)] = V$$

- $x(0)$ is a Gaussian random vector independent of $w(k)$ and $v(k)$, with

$$\mathrm{E}[x(0)] = x_0,\ \mathrm{E}[(x(0)-x_0)(x(0)-x_0)^T] = X_0$$

LQ when there is noise

Assume all states are accessible in the plant

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

The original LQ cost

$$2J = x^T(N)Sx(N) + \sum_{j=0}^{N-1}\left\{x^T(j)Qx(j) + u^T(j)Ru(j)\right\}$$

is no longer valid due to the noise term $w(k)$.
Instead, consider a stochastic performance index:

$$J = \mathop{\mathrm{E}}_{\{x(0),w(0),\ldots,w(N-1)\}}\left\{x^T(N)Sx(N) + \sum_{j=0}^{N-1}[x^T(j)Qx(j) + u^T(j)Ru(j)]\right\}$$

with $S \succeq 0,\ Q \succeq 0,\ R \succ 0$

LQ with noise and exactly known states

solution via stochastic dynamic programming:

Define "cost to go":

$$J_k(x(k)) \triangleq \underset{W_k^+}{\mathrm{E}} \left\{ x^T(N)Sx(N) + \sum_{j=k}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\},$$

$$W_k^+ = \{w(k), \ldots, w(N-1)\}$$

We look for the optima under control $U_k^+ = \{u(k), \ldots, u(N-1)\}$:

$$J_k^o(x(k)) = \min_{U_k^+} J_k(x(k))$$

- the ultimate optimal cost is

$$J^o = \underset{x(0)}{\mathrm{E}} \left[\min_{U_0^+} J_0(x(0)) \right]$$

LQ with noise and exactly known states

solution via stochastic dynamic programming:

iteration on *optimal* cost to go:

$$J_k^o(x(k)) = \min_{U_k^+} \underset{W_k^+}{\mathrm{E}} \left\{ x^T(N)Sx(N) + x^T(k)Qx(k) + u^T(k)Ru(k) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\}$$

$$= \min_{U_{k+1}^+} \min_{u(k)} \underset{W_k^+}{\mathrm{E}} \left\{ x^T(N)Sx(N) + x^T(k)Qx(k) + u^T(k)Ru(k) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right\} \quad (1)$$

$$= \min_{U_{k+1}^+} \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \underset{W_k^+}{\mathrm{E}} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (2)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \min_{U_{k+1}^+} \underset{w(k)}{\mathrm{E}} \underset{W_{k+1}^+}{\mathrm{E}} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (3)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \underset{w(k)}{\mathrm{E}} \min_{U_{k+1}} \underset{W_{k+1}^+}{\mathrm{E}} \left[x^T(N)Sx(N) + \sum_{j=k+1}^{N-1} [x^T(j)Qx(j) + u^T(j)Ru(j)] \right] \right\} \quad (4)$$

$$= \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \underset{w(k)}{\mathrm{E}} \left[J_{k+1}^o(x(k+1)) \right] \right\} \quad (5)$$

- (1) to (2): $x(k)$ does not depend on $w(k)$, $w(k+1)$,..., $w(N-1)$

LQ with noise and exactly known states

solution via stochastic dynamic programming: induction

$$J_k^o(x(k)) = \min_{u(k)} \left\{ x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \left[J_{k+1}^o(x(k+1)) \right] \right\}$$

at time N: $$J_N^o(x(N)) = x^T(N)Sx(N)$$

assume at time $k+1$:

$$J_{k+1}^o(x(k+1)) = \underbrace{x^T(k+1)P(k+1)x(k+1)}_{\text{cost in a standard LQ}} + \underbrace{b(k+1)}_{\text{due to noise}}$$

then at time k:

$$J_k^o(x(k)) = \min_{u(k)} \left(x^T(k)Qx(k) + u^T(k)Ru(k) + \mathop{\mathrm{E}}_{w(k)} \left[x^T(k+1)P(k+1)x(k+1) + b(k+1) \right] \right)$$

next: use system dynamics $x(k+1) = Ax(k) + Bu(k) + B_w w(k)$...

LQ with noise and exactly known states

after some algebra:

$$J_k^o(x(k)) = \mathop{\mathrm{E}}_{w(k)} \min_{u(k)} \{ x^T(k) \left[Q + A^T P(k+1) A \right] x(k)$$
$$+ u^T(k) \left[R + B^T P(k+1) B \right] u(k) + 2x^T(k) A^T P(k+1) Bu(k) + 2x^T(k) A^T P(k+1) B_w w(k)$$
$$+ 2u^T(k) B^T P(k+1) B_w w(k) + w(k)^T B_w^T P(k+1) B_w w(k) + b(k+1) \}$$

$w(k)$ is white and zero mean $\Rightarrow$:

$$\mathop{\mathrm{E}}_{w(k)} \left\{ 2x^T(k) A^T P(k+1) B_w w(k) + 2u^T(k) B^T P(k+1) B_w w(k) \right\} = 0$$

$\mathop{\mathrm{E}}_{w(k)} \left\{ w(k)^T B_w^T P(k+1) B_w w(k) \right\}$ equals

$$\mathrm{Tr} \left\{ \mathop{\mathrm{E}}_{w(k)} \left[B_w^T P(k+1) B_w w(k) w(k)^T \right] \right\} = \mathrm{Tr} \left[B_w^T P(k+1) B_w W \right]$$

other terms: not random w.r.t. $w(k)$; can be taken outside of $\mathrm{E}_{w(k)}$

LQ with noise and exactly known states

therefore

$$J_k^o(x(k)) = \min_{u(k)} \{x^T(k) \left[Q + A^T P(k+1) A\right] x(k)$$
$$+ u^T(k) \left[R + B^T P(k+1) B\right] u(k) + 2x^T(k) A^T P(k+1) B u(k)\}$$
$$+ \mathrm{Tr}\left[B_w^T P(k+1) B_w W\right] + b(k+1)$$

note: the term inside the minimization is a quadratic (actually convex) function of $u(k)$. Optimization is easily done.

Recall: facts of quadratic functions

- consider

$$f(u) = \frac{1}{2} u^T M u + p^T u + q, \; M = M^T \tag{6}$$

- optimality (maximum when M is negative definite; minimum when M is positive definite) is achieved when

$$\frac{\partial f}{\partial u^o} = M u^o + p = 0 \Rightarrow u^o = -M^{-1} p \tag{7}$$

- and the optimal cost is

$$f^o = f(u^o) = -\frac{1}{2} p^T M^{-1} p + q \tag{8}$$

LQ with noise and exactly known states

$$J_k^o(x(k)) = \min_{u(k)} \left\{ u^T(k) \left[R + B^T P(k+1) B \right] u(k) + 2x^T(k) A^T P(k+1) Bu(k) + x^T(k) \left[Q + A^T P(k+1) A \right] x(k) \right\} + \text{Tr}\left[B_w^T P(k+1) B_w W \right] + b(k+1)$$

- optimal control law [by using (7)]:

$$\boxed{u^o(k) = -\left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) Ax(k)}$$

- optimal cost [by using (8)]:

$$J_k^o(x(k)) = \left\{ -x^T(k) A^T P(k+1) B \left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) Ax(k) + x^T(k) \left[Q + A^T P(k+1) A \right] x(k) \right\} + \text{Tr}\left[B_w^T P(k+1) B_w W \right] + b(k+1)$$

LQ with noise and exactly known states

Riccati equation:

the optimal cost

$$J_k^o(x(k)) = \left\{ -x^T(k) A^T P(k+1) B \left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) Ax(k) + x^T(k) \left[Q + A^T P(k+1) A \right] x(k) \right\} + \text{Tr}\left[B_w^T P(k+1) B_w W \right] + b(k+1)$$

can be written as

$$\boxed{J_k^o(x(k)) = x^T(k) P(k) x(k) + b(k)}$$

with the Riccati equation

$$\boxed{P(k) = A^T P(k+1) A - A^T P(k+1) B \left[R + B^T P(k+1) B \right]^{-1} B^T P(k+1) A + Q}$$

and

$$\boxed{b(k) = \text{Tr}\left[B_w^T P(k+1) B_w W \right] + b(k+1)}$$

boundary conditions: $P(N) = S$ and $b(N) = 0$

LQ with noise and exactly known states

observations:

- optimal control law and Riccati equation are the same as those in the regular LQ problem
- addition cost is due to $B_w w(k)$:

$$b(k) = \text{Tr}\left[B_w^T P(k+1) B_w W\right] + b(k+1),\ b(N) = 0$$

- the final optimal cost is

$$\begin{aligned} J^o(x(0)) &= \mathop{\text{E}}_{x(0)}\left[x^T(0)P(0)x(0) + b(0)\right] \\ &= \mathop{\text{E}}_{x(0)}\left[(x_o + x(0) - x_o)^T P(0)(x_o + x(0) - x_o) + b(0)\right] \\ &= x_o^T P(0) x_o + \text{Tr}(P(0)X_o) + b(0) \end{aligned} \quad (9)$$

where

$$b(0) = \sum_{j=0}^{N-1} \text{Tr}\left[B_w^T P(j+1) B_w W\right]$$

LQG: LQ with noise and inexactly known states

notice that

- not all states may be available and there is usually output noise:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k) \\ y(k) &= Cx(k) + v(k) \end{aligned}$$

- when u is a function of y, the cost has to also consider the randomness from $V_k^+ = \{v(k), \dots, v(N-1)\}$

$$\boxed{J = \mathop{\text{E}}_{x(0), W_0^+, V_0^+}\left\{x^T(N)Sx(N) + \sum_{j=0}^{N-1}[x^T(j)Qx(j) + u^T(j)Ru(j)]\right\}} \quad (10)$$

these motivate the linear quadratic Gaussian (LQG) control problem

LQG solution

only $y(k)$ is accessible instead of $x(k)$, some connection has to be built to connect the cost to $Y_k = \{y(0), \dots, y(k)\}$:

$$
\begin{aligned}
& \mathrm{E}\left[x^T(k) Q x(k)\right] \\
=& \mathrm{E}\left\{\mathrm{E}\left[x^T(k) Q x(k) \middle| Y_k\right]\right\} \\
=& \mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k)+\hat{x}(k|k))^T Q (x(k)-\hat{x}(k|k)+\hat{x}(k|k)) \middle| Y_k\right]\right\} \\
=& \mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q (x(k)-\hat{x}(k|k)) \middle| Y_k + \hat{x}^T(k|k) Q \hat{x}(k|k) \middle| Y_k \right.\right. \\
& \qquad \left.\left. +2(x(k)-\hat{x}(k|k))^T Q \hat{x}(k|k) \middle| Y_k\right]\right\} \qquad (11)
\end{aligned}
$$

LQG solution

but $\mathrm{E}[x(k)|Y_k] = \hat{x}(k|k)$ and $\hat{x}(k|k)$ is orthogonal to $\tilde{x}(k|k)$ (property of least square estimation), so

$$
\begin{aligned}
\mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q \hat{x}(k|k) \middle| Y_k\right]\right\} &= \mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q \hat{x}(k|k)\right] \\
&= \mathrm{Tr}\mathrm{E}\left[Q \hat{x}(k|k) \tilde{x}^T(k|k)\right] = 0
\end{aligned}
$$

yielding

$$
\begin{aligned}
& \underline{\mathrm{E}\left[x^T(k) Q x(k)\right]} \\
=& \mathrm{E}\left\{\mathrm{E}\left[(x(k)-\hat{x}(k|k))^T Q (x(k)-\hat{x}(k|k)) \middle| Y_k + \hat{x}^T(k|k) Q \hat{x}(k|k) \middle| Y_k\right]\right\} \\
=& \mathrm{E}\left[\hat{x}^T(k|k) Q \hat{x}(k|k) \middle| Y_k\right] \\
& + \mathrm{E}\left\{\mathrm{E}\left[\mathrm{Tr}\left\{Q (x(k)-\hat{x}(k|k))(x(k)-\hat{x}(k|k))^T\right\} \middle| Y_k\right]\right\} \\
=& \underline{\mathrm{E}\left[\hat{x}^T(k|k) Q \hat{x}(k|k)\right] + \mathrm{Tr}\{Q Z(k)\}}
\end{aligned}
$$

LQG solution

the LQG cost (10) is thus

$$J = \overbrace{\mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)]\right\}}^{\hat{J}}$$

$$+ \underbrace{\mathrm{Tr}\{SZ(N)\} + \sum_{j=0}^{N-1}\mathrm{Tr}\{QZ(j)\}}_{\text{independent of the control input}}$$

hence

$$\boxed{\min_{\{u(0),\ldots,u(N-1)\}} J \Longleftrightarrow \min_{\{u(0),\ldots,u(N-1)\}} \hat{J}}$$

LQG is equivalent to an LQ with exactly know states

consider the equivalent problem to minimize:

$$\hat{J} = \mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)]\right\}$$

- $\hat{x}(k|k)$ is fully accessible, with the dynamics:

$$\begin{aligned}\hat{x}(k+1|k+1) &= \hat{x}(k+1|k) + F(k+1)e_y(k+1)\\ &= A\hat{x}(k|k) + Bu(k) + F(k+1)e_y(k+1)\end{aligned}$$

- from KF results, $e_y(k+1)$ is white, Gaussian with covariance:

$$V + CM(k+1)C^T$$

LQG is equivalent to LQ with exactly know states

LQ with exactly known states:

$$J = \mathrm{E}\left\{x^T(N)Sx(N) + \sum_{j=0}^{N-1}[x^T(j)Qx(j) + u^T(j)Ru(j)]\right\}$$

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

$$u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) Ax(k)$$

LQG:

$$\hat{J} = \mathrm{E}\left\{\hat{x}^T(N|N)S\hat{x}(N|N) + \sum_{j=0}^{N-1}[\hat{x}^T(j|j)Q\hat{x}(j|j) + u^T(j)Ru(j)]\right\}$$

$$\hat{x}(k+1|k+1) = A\hat{x}(k|k) + Bu(k) + F(k+1)e_y(k+1)$$

the solution of LQG is thus:

$$\boxed{u^o(k) = -\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A\hat{x}(k|k)} \qquad (12)$$

$$\boxed{P(k) = A^T P(k+1) A - A^T P(k+1) B\left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A + Q}$$

Optimal cost of LQG control

- LQ with known states (see (9)):

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

$$J^o = x_o^T P(0) x_o + \mathrm{Tr}(P(0) X_o) + \underbrace{\sum_{j=0}^{N-1} \mathrm{Tr}\left[B_w^T P(j+1) B_w W\right]}_{b(0)}$$

- LQG:

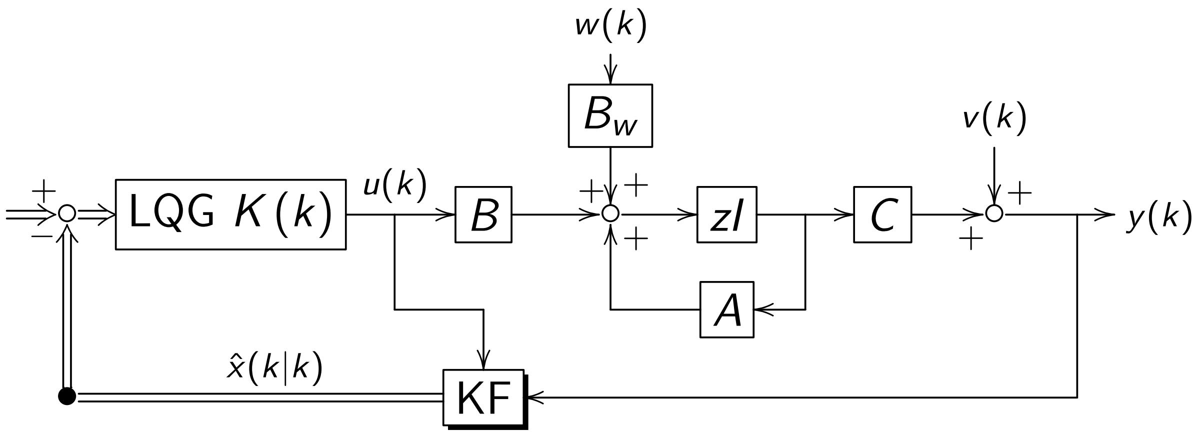
$$\hat{x}(k+1|k+1) = A\hat{x}(k|k) + Bu(k) + F(k+1)e_y(k+1)$$

$$\hat{J}^o = x_o^T P(0) x_o + \mathrm{Tr}[P(0)Z(0)]$$
$$+ \sum_{j=0}^{N-1} \mathrm{Tr}\left\{F^T(j+1)P(j+1)F(j+1)[V + CM(k+1)C^T]\right\} \qquad (13)$$

$$J_o = \hat{J}_o + \sum_{j=0}^{N-1} \mathrm{Tr}\{QZ(j)\} + \mathrm{Tr}\{SZ(N)\}$$

Separation theorem in LQG

KF: an (optimal) **observer**
LQ: an (optimal) **state feedback** control
Separation theorem in observer state feedback holds–the closed-loop dynamics contains two **separated** parts: LQ dynamics plus KF dynamics



Stationary LQG problem

Assumptions: system is time invariant; weighting matrices in performance index is time-invariant; noises are white, Gaussian, wide sense stationary.
Equivalent problem: minimize

$$J^{'} = \lim_{N\to\infty} \frac{J}{N} = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{x^T(N)Sx(N)}{N} + \frac{1}{N}\sum_{j=0}^{N-1}\left[x^T(j)Qx(j) + u^T(j)Ru(j)\right] \right\}$$

$$= \mathrm{E}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right]$$

Solution of stationary LQG problem

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$
$$J' = \mathrm{E}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right]$$

the solution is $u = -K_s \hat{x}(k|k)$: steady-state LQ + steady-state KF

$$K_s = \left[R + B^T P_s B\right]^{-1} B^T P_s A$$
$$P_s = A^T P_s A - A^T P_s B\left[R + B^T P_s B\right]^{-1} B^T P_s A + Q$$
$$F_s = M_s C^T \left[C M_s C^T + V\right]^{-1}$$
$$M_s = A M_s A^T - A M_s C^T \left[C M_s C^T + V\right]^{-1} C M_s A^T + B_w W B_w^T$$

stability and convergence conditions of the Riccati equations:

- (A, B_w) and (A, B): controllable or stabilizable
- (A, C_q) and (A, C): observable or detectable $(Q = C_q^T C_q)$

Solution of stationary LQG problem

- stability conditions: guaranteed closed-loop stability and KF stability
- separation theorem: closed-loop eigenvalues come from
  - the n eigenvalues of LQ state feedback: $A - BK_s$
  - the n eigenvalues of KF: $A - AF_s C$ (or equivalently $A - F_S CA$)
- optimal cost:

$$J_\infty^o = \mathrm{Tr}\left[P_s\left(BK_s Z_s A^T + B_w W B_w^T\right)\right] \qquad (14)$$

- exercise: prove (14)

Continuous-time LQG

- plant:

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)+B_w w(t)\\ y(t) &= Cx(t)+v(t)\end{aligned}$$

- assumptions: $w(t)$ and $v(t)$ are Gaussian and white; $x(0)$ is Gaussian
- cost:

$$J=\mathrm{E}\left\{x^T(t_f)\,Sx(t_f)+\int_{t_0}^{t_f}\left[x^T(t)\,Q(t)\,x(t)+u^T(t)\,R(t)\,u(t)\right]\mathrm{d}t\right\}$$

where $S\succeq 0$, $Q(t)\succeq 0$, and $R(t)\succ 0$ and the expectation is taken over all random quantities $\{x(0),w(t),v(t)\}$

Continuous-time LQG solution

- Continuous-time LQ:

$$u(t)=-R^{-1}B^TP(t)\hat{x}(t|t) \tag{15}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t}=A^TP+PA-PBR^{-1}B^TP+Q,\ \ P(t_f)=S \tag{16}$$

- Continuous-time KF:

$$\frac{\mathrm{d}\hat{x}(t|t)}{\mathrm{d}t}=A\hat{x}(t|t)+Bu(t)+F(t)\left(y(t)-C\hat{x}(t|t)\right) \tag{17}$$

$$F(t)=M(t)C^TV^{-1},\ \ \hat{x}(t_0|t_0)=x_o \tag{18}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t}=AM+MA^T-MC^TV^{-1}CM+B_wWB_W^T,\ \ M(t_0)=X_o \tag{19}$$

Summary

1. Big picture
2. Stochastic control with exactly known state
3. Stochastic control with inexactly known state
4. Steady-state LQG
5. Continuous-time LQG problem