Lecture 10: LQ with Frequency Shaped Cost Function (FSLQ)

Big picture

why are we learning this:

- in standard LQ, Q and R are constant matrices in the cost function

$$J = \int_0^\infty \left(x^T(t) Q x(t) + \rho u^T(t) R u(t) \right) \mathrm{d}t \tag{1}$$

- how can we introduce more design freedom for Q and R?

Connection between time and frequency domains

Theorem (Parseval's Theorem)

For a square integrable signal $f(t)$ defined on $[0, \infty)$

$$\int_0^{\infty} f^T(t) f(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^T(-j\omega) F(j\omega)\, d\omega$$

1D case:

$$\int_0^{\infty} |f(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \mathrm{d}\omega$$

Intuition: energy in time-domain equals energy in frequency domain

For the general case, $f(t)$ can be acausal. We have

$$\int_{-\infty}^{\infty} f^T(t) f(t)\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^T(-j\omega) F(j\omega)\, \mathrm{d}\omega$$

Discrete-time version:

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^T\left(e^{-j\omega}\right) F\left(e^{j\omega}\right) \mathrm{d}\omega$$

History

Marc-Antoine Parseval (1755-1836):

- French mathematician
- published just five (but important) mathematical publications in total (source: Wikipedia.org)

Frequency-domain LQ cost function

From Parseval's Theorem, the LQ cost in frequency domain is

$$J = \int_0^\infty \left(x^T(t) Q x(t) + \rho u^T(t) R u(t) \right) \mathrm{d}t \tag{2}$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty \left(X^T(-j\omega) Q X(j\omega) + \rho U^T(-j\omega) R U(j\omega) \right) \mathrm{d}\omega \tag{3}$$

Frequency-shaped LQ expands Q and R to frequency-dependent functions:

$$J = \frac{1}{2\pi} \int_{-\infty}^\infty \left(X^T(-j\omega) Q(j\omega) X(j\omega) + \rho U^T(-j\omega) R(j\omega) U(j\omega) \right) \mathrm{d}\omega \tag{4}$$

Frequency-domain LQ cost function

Let

$$Q(j\omega) = Q_f^T(-j\omega) Q_f(j\omega) \succeq 0, \; X_f(j\omega) = Q_f(j\omega) X(j\omega)$$
$$R(j\omega) = R_f^T(-j\omega) R_f(j\omega) \succ 0, \; U_f(j\omega) = R_f(j\omega) U(j\omega)$$

(4) becomes

$$J = \frac{1}{2\pi} \int_{-\infty}^\infty \left(X_f^T(-j\omega) X_f(j\omega) + \rho U_f^T(-j\omega) U_f(j\omega) \right) \mathrm{d}\omega$$

which is equivalent to (using Parseval's Theorem again)

$$\boxed{J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho u_f^T(t) u_f(t) \right) \mathrm{d}t} \tag{5}$$

Frequency-domain LQ cost function

Summarizing, we have:

- plant:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{6}$$

- new cost:

$$J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho u_f^T(t) u_f(t) \right) \mathrm{d}t \tag{7}$$

- filtered states and inputs:

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)\,, \quad u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

We just need to translate the problem to a standard one [which we know (very well) how to solve]

Frequency-domain weighting filters

state filtering

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)$$

- a MIMO process in general: if $x(t) \in \mathbb{R}^n$ and $x_f(t) \in \mathbb{R}^q$, then $Q_f(s)$ is a $q \times n$ transfer function matrix
- $Q_f(s)$: state filter; designer's choice; can be selected to meet the desired control action and the performance requirements

- write $Q_f(s) = C_1 (sI - A_1)^{-1} B_1 + D_1$ in the general state-space realization:

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 x(t) \\ x_f(t) = C_1 z_1(t) + D_1 x(t) \end{cases} \tag{8}$$

Frequency-domain weighting filters

input filtering

$$u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

- $R_f(s)$: input filter; designer's choice; can be selected to meet the robustness requirements
- write $R_f(s) = C_2(sI - A_2)^{-1}B_2 + D_2$ in the general state-space realization:

$$\begin{cases} \dot{z}_2(t) &= A_2 z_2(t) + B_2 u(t) \\ u_f(t) &= C_2 z_2(t) + D_2 u(t) \end{cases} \tag{9}$$

Back to time-domain design

Combining (6), (8) and (9) gives the enlarged system

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}}_{x_e(t)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(t)$$

and

$$x_f(t) = \underbrace{[D_1 \;\; C_1 \;\; 0]}_{C_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$$

$$u_f(t) = [0 \;\; 0 \;\; C_2]x_e(t) + D_2 u(t)$$

Summary of solution

With the enlarged system, the cost

$$J=\int_0^\infty\left(x_f^T(t)x_f(t)+\rho u_f^T(t)u_f(t)\right)\mathrm{d}t \tag{10}$$

translates to

$$J=\int_0^\infty\left(x_e^T(t)Q_ex_e(t)+2u^T(t)\underbrace{\begin{bmatrix}0 & 0 & \rho D_2^TC_2\end{bmatrix}}_{N_e}x_e(t)+u^T(t)\underbrace{\rho D_2^TD_2}_{R_e}u(t)\right)\mathrm{d}t$$

$$Q_e=\begin{bmatrix}D_1^TD_1 & D_1^TC_1 & 0\\ C_1^TD_1 & C_1^TC_1 & 0\\ 0 & 0 & \rho C_2^TC_2\end{bmatrix}$$

- solution (see appendix for more details):

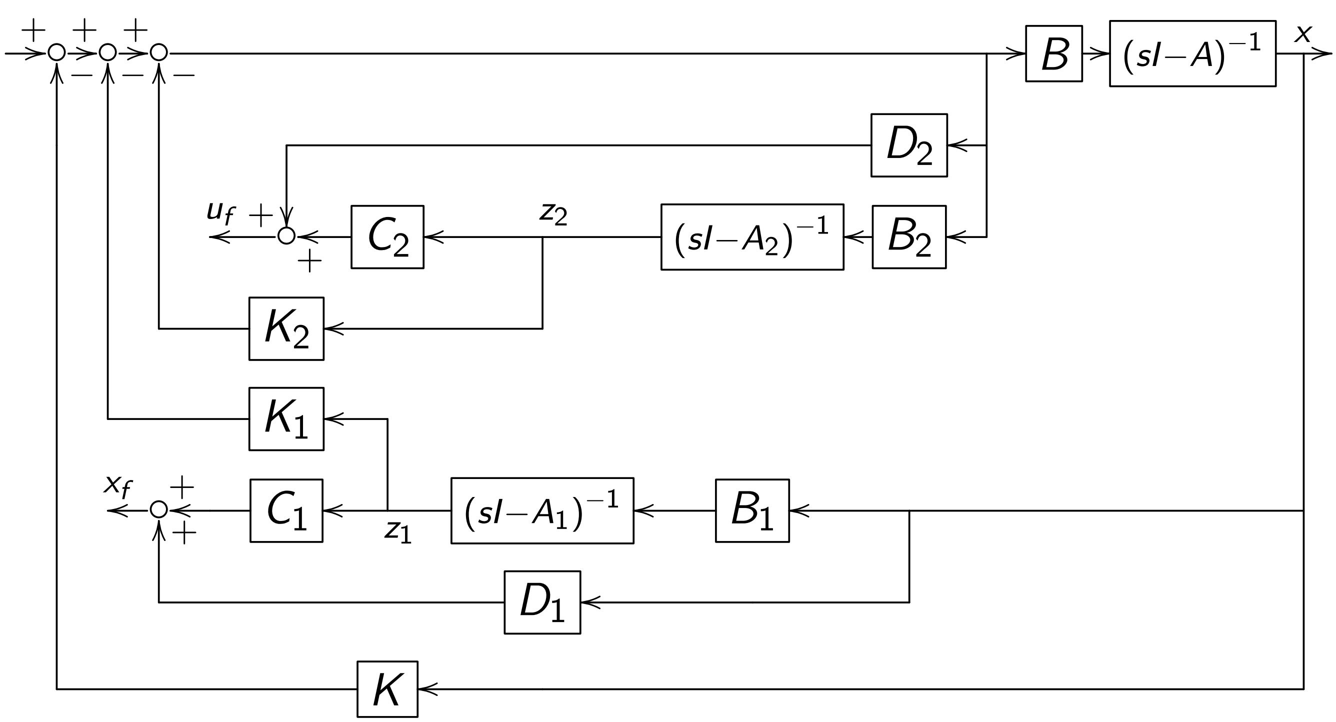
$$\boxed{u(t)=-R_e^{-1}(B_e^TP_e+N_e)x_e(t)=-Kx(t)-K_1z_1(t)-K_2z_2(t)}$$

- algebraic Riccati equation:

$$\boxed{A_e^TP_e+P_eA_e-(B_e^TP_e+N_e)^TR_e^{-1}(B_e^TP_e+N_e)+Q_e=0}$$

Implementation

structure of the FSLQ system:



Appendix: general LQ solution

Consider LQ problems with cost

$$J = \int_0^\infty \left(x^T(t) \underbrace{C^T C}_{Q} x(t) + 2u^T(t) N x(t) + u^T(t) R u(t) \right) \mathrm{d}t \quad (11)$$

and system dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- assume (A, B) is controllable/stabilizable and (A, C) is observable/detectable
- the solution of the problem is

$$u(t) = -R^{-1}(B^T P + N)x(t)$$

$$A^T P + PA - (B^T P + N)^T R^{-1}(B^T P + N) + Q = 0$$

Appendix: general LQ solution

Intuition: under the assumptions, we know we can stabilize the system and drive $x(t)$ to zero. Consider $V(t) = x^T(t) P x(t)$

$$\cancelto{0}{V(\infty)} - V(0) = \int_0^\infty \dot{V}(t)\mathrm{d}t$$
$$= \int_0^\infty \left(x^T(t) \left(PA + A^T P \right) x(t) + 2x^T(t) PBu(t) \right) \mathrm{d}t$$

Adding (11) on both sides yields

$$V(\infty) - V(0) + J =$$
$$\int_0^\infty \left(x^T(t) \left(Q + PA + A^T P \right) x(t) + 2x^T(t) \left(PB + N^T \right) u(t) + u^T(t) R u(t) \right) \mathrm{d}t \quad (12)$$

- to minimize the cost, we are going to re-organize the terms in (12) into some "squared" terms

Appendix: general LQ solution

"completing the squares":

$$2x^T(t)\left(PB+N^T\right)u(t)+u^T(t)Ru(t)=\left\|R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2$$
$$-x^T(t)\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)x(t)$$

hence (12) is actually

$$\cancelto{0}{V(\infty)}-V(0)+J$$
$$=\int_0^\infty\left[x^T(t)\left(Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)\right)x(t)\right.$$
$$\left.+\left\|R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2\right]\mathrm{d}t$$

hence $J_{\min}=V(0)=x^T(0)Px(0)$ is achieved when

$$Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)=0$$

$$\text{and } u(t)=-R^{-1}\left(B^TP+N\right)x(t)$$