Principles of Feedback Design

MIMO closed-loop analysis
Robust stability
MIMO feedback design

Big picture

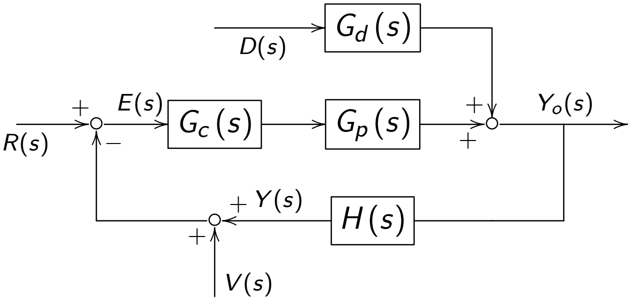
- we are pretty familiar with SISO feedback system design and analysis
- state-space designs (LQ, KF, LQG,...): time-domain; good mathematical formulation and solutions based on rigorous linear algebra
- frequency-domain and transfer-function analysis: builds intuition; good for properties such as stability robustness

MIMO closed-loop analysis

signals and transfer functions are vectors and matrices now:

- r (reference) and y (plant output): m-dimensional
- $G_p(s)$: p by m transfer function matrix

$$\begin{aligned} E(s) &= R(s) - (H(s)\,Y_o(s) + V(s)) \\ &= R(s) - \{H(s)\,G_p(s)\,G_c(s)\,E(s) + H(s)\,G_d(s)\,D(s) + V(s)\} \end{aligned} \quad (1)$$



MIMO closed-loop analysis

(1) gives

$$E(s) = (I_m + G_{\text{open}}(s))^{-1} R(s)$$
$$- (I_m + G_{\text{open}}(s))^{-1} H(s)\, G_d(s)\, D(s) - (I_m + G_{\text{open}}(s))^{-1} V(s)$$

where the *loop transfer function*

$$\boxed{G_{\text{open}}(s) = H(s)\, G_p(s)\, G_c(s)}$$

We want to minimize $E^*(s) \triangleq R(s) - Y(k) = E(s) + V(s)$

$$E^*(s) = \underline{(I_m + G_{\text{open}}(s))^{-1}} R(s)$$
$$- (I_m + G_{\text{open}}(s))^{-1} H(s)\, G_d(s)\, D(s) + \underline{(I_m + G_{\text{open}}(s))^{-1} G_{\text{open}}(s)} V(s)$$

Sensitivity and complementary sensitivity functions:

$$S(s) \triangleq (I_m + G_{\text{open}}(s))^{-1}$$
$$T(s) \triangleq (I_m + G_{\text{open}}(s))^{-1} G_{\text{open}}(s)$$

Fundamental limitations in feedback design

$$E^*(s) = S(s)R(s) + T(s)V(s) - S(s)H(s)G_d(s)D(s)$$
$$Y(s) = R(s) - E^*(s) = T(s)R(s) + \dots$$

- sensitivity function $S(s)$: explains disturbance-rejection ability
- complementary sensitivity function $T(s)$: explains reference tracking and sensor-noise rejection abilities
- fundamental constraint of feedback design:

$$\boxed{S(s) + T(s) = I_m}$$

  equivalently

$$S(j\omega) + T(j\omega) = I_m$$

- cannot do well in all aspects: e.g., if $S(j\omega) \approx 0$ (good disturbance rejection), $T(j\omega)$ will be close to identity (bad sensor-noise rejection)

Goals of SISO control design

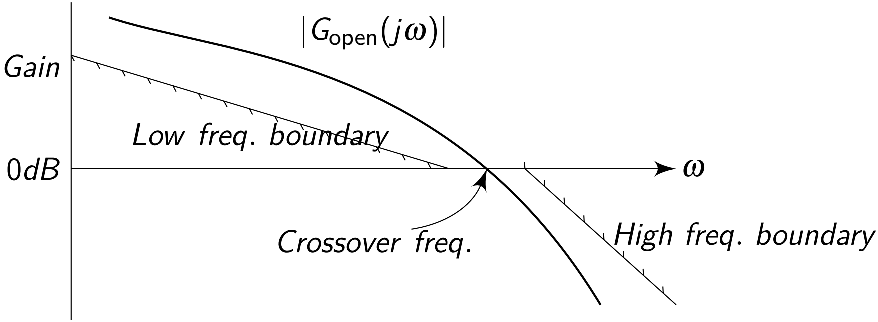
single-input single-output (SISO) control design:

$$S(j\omega) = \frac{1}{1 + G_{\text{open}}(j\omega)}, \ T(j\omega) = \frac{G_{\text{open}}(j\omega)}{1 + G_{\text{open}}(j\omega)}$$

- goals:
  1. nominal stability
  2. stability robustness
  3. command following and disturbance rejection
  4. sensor-noise rejection
- feedback achieves: 1 (Nyquist theorem), 2 (sufficient (gain and phase) margins), and
  - 3: small $S(j\omega)$ at relevant frequencies (usually low frequency)
  - 4: small $T(j\omega)$ at relevant frequencies (usually high frequency)
- additional control design for meeting the performance goals: feedforward, predictive, preview controls, etc

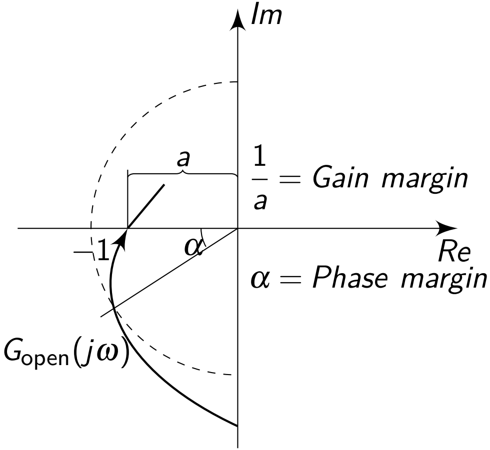
SISO loop shaping

typical loop shape (magnitude response of G_{open}):



SISO loop shaping: stability robustness

the idea of stability margins:



SISO loop shaping: stability robustness

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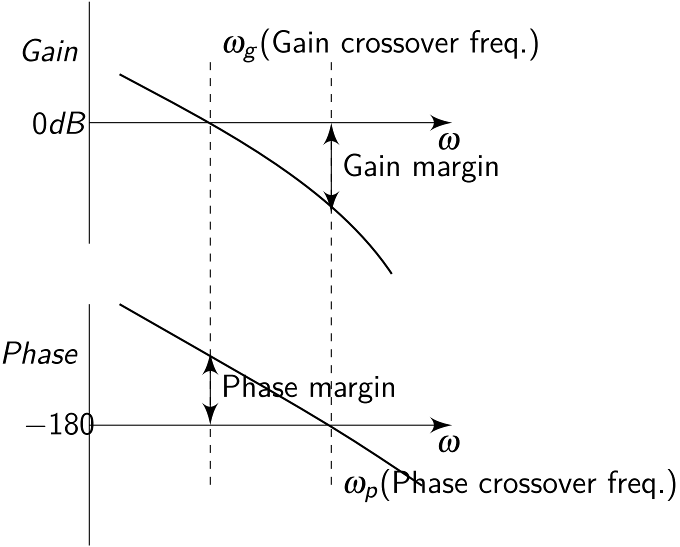
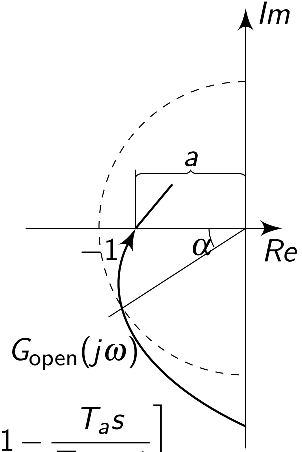


SISO loop shaping: stability robustness

$G_{\text{open}}(j\omega)$ should be sufficiently far away from $(-1,0)$ for robust stability.
Commonly there are uncertainties and the actual case is

$$\tilde{G}_{\text{open}}(s) = G_{\text{open}}(s)\left[1+\Delta(s)\right]$$

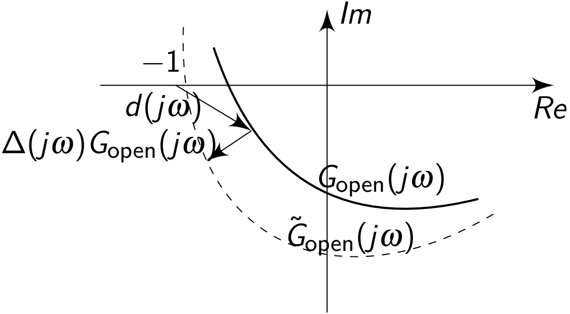
e.g. ignored actuator dynamics in a positioning system:



$$\tilde{G}_{\text{open}}(s) = G_{\text{open}}(s)\frac{1}{T_a s+1} = G_{\text{open}}(s)\left[1-\frac{T_a s}{T_a s+1}\right]$$

$$\Delta(j\omega) = -\frac{T_a j\omega}{T_a j\omega+1}$$

SISO loop shaping: stability robustness



if nominal stability holds, robust stability needs

$$|\Delta(j\omega)\, G_{open}(j\omega)| = \left|\tilde{G}_{open}(j\omega) - G_{open}(j\omega)\right| < \overbrace{|1 + G_{open}(j\omega)|}^{|d(j\omega)|}$$

$$\Leftrightarrow \boxed{\left|\Delta(j\omega)\,\frac{G_{open}(j\omega)}{1 + G_{open}(j\omega)}\right| < 1 \Leftrightarrow |\Delta(j\omega)\, T(j\omega)| < 1,\ \forall\omega}$$

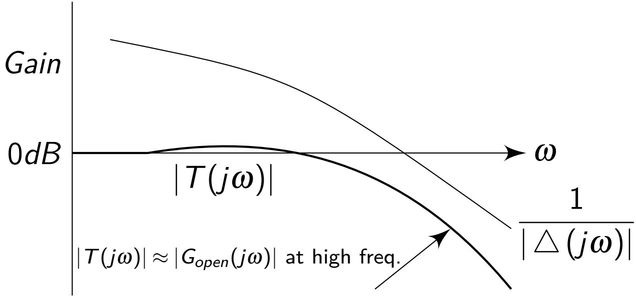
SISO loop shaping: stability robustness

if $|G_{open}(j\omega)| \ll 1$ then

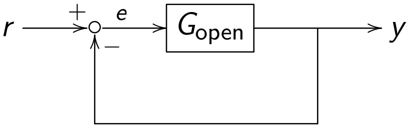
$$\boxed{\left|\Delta(j\omega)\frac{G_{open}(j\omega)}{1+G_{open}(j\omega)}\right| < 1}$$

approximately means

$$|G_{open}(j\omega)| < \frac{1}{|\Delta(j\omega)|}$$



MIMO Nyquist criterion



- assume G_{open} is $m \times m$ and realized by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Be(t),\ x \in \mathbb{R}^{m\times 1}$$
$$y(t) = Cx(t)$$

- the closed-loop dynamics is

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = (A - BC)x(t) + Br(t) \\ y(t) & = Cx(t) \end{cases} \tag{2}$$

MIMO Nyquist criterion

(2) gives the closed-loop transfer function

$$G_{\text{closed}}(s) = C\left(sI - A + BC\right)^{-1} B$$

- closed-loop stability depends on the eigenvalues $\operatorname{eig}(A - BC)$, which come from

$$\begin{aligned}
\phi_{\text{closed}}(s) &= \det(sI - A + BC) = \det\left\{(sI - A)\left[I + (sI - A)^{-1} BC\right]\right\} \\
&= \det(sI - A)\det\left(I + C(sI - A)^{-1} B\right) \\
&= \underbrace{\det(sI - A)}_{\text{open loop } \phi_{\text{open}}(s)} \det\left(I + G_{\text{open}}(s)\right)
\end{aligned}$$

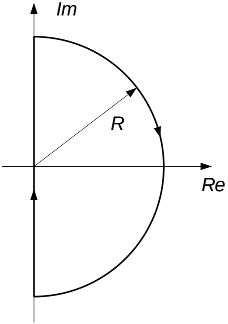
- hence

$$\boxed{\frac{\phi_{\text{closed}}(s)}{\phi_{\text{open}}(s)} = \det\left(I + G_{\text{open}}(s)\right)}$$

MIMO Nyquist criterion

$$\boxed{\frac{\phi_{\text{closed}}(s)}{\phi_{\text{open}}(s)} = \det\left(I + G_{\text{open}}(s)\right) = \frac{\prod_{j=1}^{n_1}(s - p_{\text{cl}})}{\prod_{i=1}^{n_2}(s - p_{\text{ol}})}}$$

- evaluate $\det\left(I + G_{\text{open}}(s)\right)$ along the D contour $(R \to \infty)$
- Z closed-loop "unstable" eigen values in $\prod_{j=1}^{n_1}(s - p_{\text{cl}})$ contribute to $2\pi Z$ net increase in phase
- P open-loop "unstable" eigen values in $\prod_{j=1}^{n_2}(s - p_{\text{ol}})$ contribute to $-2\pi P$ net increase in phase
- stable eigen values do not contribute to net phase change



MIMO Nyquist criterion

the number of counter clockwise encirclements of the origin by $\det(I + G_{\text{open}}(s))$ is:

$$\boxed{\mathrm{N}\left(0, \det\left(I + G_{\text{open}}(s)\right), \mathrm{D}\right) = P - Z}$$

stability condition: $Z = 0$

Theorem (Multivariable Nyquist Stability Criterion)

the closed-loop system is asymptotically stable if and only if

$$\mathrm{N}\left(0, \det\left(I + G_{\text{open}}(s)\right), \mathrm{D}\right) = P$$

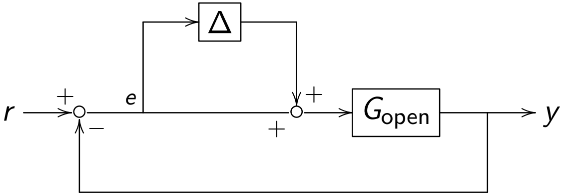
i.e., the number of counterclockwise encirclements of the origin by $\det(I + G_{\text{open}}(s))$ *along the* D *contour equals the number of open-loop unstable eigen values (of the A matrix).*

MIMO robust stability

Given the nominal model G_{open}, let the actual open loop be perturbed to

$$\tilde{G}_{\text{open}}\,(j\omega) = G_{\text{open}}(j\omega)\left[I + \Delta(j\omega)\right]$$

where $\Delta(j\omega)$ is the uncertainty (bounded by $\sigma(\Delta(j\omega)) \leq \bar{\sigma}$)



- what properties should the nominal system possess in order to have robust stability?

MIMO robust stability

- obviously need a stable nominal system to start with:

$$\mathrm{N}\left(0, \det\left(I + G_{\text{open}}\left(s\right)\right), \mathrm{D}\right) = P$$

- for robust stability, we need

$$\mathrm{N}\left(0, \det\left(I + G_{\text{open}}\left(s\right)\left(1 + \Delta\left(s\right)\right)\right), \mathrm{D}\right) = P \text{ for all possible } \Delta$$

- under nominal stability, we need the boundary condition

$$\boxed{\det\left(I + G_{\text{open}}\left(j\omega\right)\left(1 + \Delta\left(j\omega\right)\right)\right) \neq 0}$$

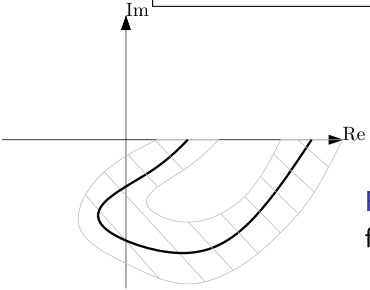


Figure: Example Nyquist plot for robust stability analysis

MIMO robust stability

- note the determinant equivalence:

$$\det\left(I + G_{\text{open}}(j\omega)\left(1 + \Delta(j\omega)\right)\right) = \det\left(I + G_{\text{open}}(j\omega)\right) \\ \times \det\left[I + \left(I + G_{\text{open}}(j\omega)\right)^{-1} G_{\text{open}}(j\omega)\,\Delta(j\omega)\right]$$

- as the system is open-loop asymptotically stable, no poles are on the imaginary, i.e.,

$$\det\left(I + G_{\text{open}}(j\omega)\right) \neq 0$$

- hence $\det\left(I + G_{\text{open}}(j\omega)\left(1 + \Delta(j\omega)\right)\right) \neq 0 \Longleftrightarrow$

$$\det\left[I + \underbrace{\left(I + G_{\text{open}}(j\omega)\right)^{-1} G_{\text{open}}(j\omega)}_{T(j\omega)}\Delta(j\omega)\right] \neq 0 \qquad (3)$$

MIMO robust stability

- intuitively, (3) means $T(j\omega)\Delta(j\omega)$ should be “smaller than” I
- mathematically, (3) will be violated if $\exists\ x \neq 0$ that achieves

$$\begin{aligned}&[I + T(j\omega)\Delta(j\omega)]x = 0\\ \Leftrightarrow\ & T(j\omega)\Delta(j\omega)x = -x\end{aligned} \tag{4}$$

  which will make the singular value

$$\sigma_{\max}[T(j\omega)\Delta(j\omega)] = \max_{v\neq 0}\frac{||T(j\omega)\Delta(j\omega)v||_2}{||v||_2} \geq \frac{||T(j\omega)\Delta(j\omega)x||_2}{||x||_2}$$

- as this cannot happen, we must have

$$\sigma_{\max}[T(j\omega)\Delta(j\omega)] < 1$$

  It turns out this is both necessary and sufficient if $\Delta(j\omega)$ is unstructured (can 'attack' from any directions). Message: we can design G_{open} such that $\sigma_{\max}[\Delta(j\omega)] < \sigma_{\min}\left[T^{-1}(j\omega)\right]$.

Summary

1. Big picture
2. MIMO closed-loop analysis
3. Loop shaping
   - SISO case
4. MIMO stability and robust stability
   - MIMO Nyquist criterion
   - MIMO robust stability