Review of Probability Theory

Connection with control systems
Random variable, distribution
Multiple random variables
Random process, filtering a random process

Big picture

why are we learning this:

We have been very familiar with deterministic systems:

$$x(k+1) = Ax(k) + Bu(k)$$

In practice, we commonly have:

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

where $w(k)$ is the noise term that we have been neglecting. With the introduction of $w(k)$, we need to equip ourselves with some additional tool sets to understand and analyze the problem.

Sample space, events and probability axioms

- experiment: a situation whose outcome depends on chance
- trial: each time we do an experiment we call that a trial

Example (Throwing a fair dice)

possible outcomes in one trail: getting a ONE, getting a TWO, ...

- sample space Ω: includes all the possible outcomes
- probability: discusses how likely things, or more formally, events, happen
- an event S_i: includes some (maybe 1, maybe more, maybe none) outcomes of the sample space. e.g., the event that it won't rain tomorrow; the event that getting odd numbers when throwing a dice

Sample space, events and probability axioms

probability axioms

- $\Pr\{S_j\} \geq 0$
- $\Pr\{\Omega\} = 1$
- if $S_i \cap S_j = \varnothing$ (empty set), then $\Pr\{S_i \cup S_j\} = \Pr\{S_i\} + \Pr\{S_j\}$

Example (Throwing a fair dice)

the sample space:

$$\Omega = \{\underbrace{\text{getting a ONE}}_{\omega_1}, \underbrace{\text{getting a TWO}}_{\omega_2}, \ldots, \underbrace{\text{getting a SIX}}_{\omega_6}\}$$

the event S_1 of observing an even number:

$$S_1 = \{\omega_2, \omega_4, \omega_6\}$$

$$\Pr\{S_1\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Random variables

to better measure probabilities, we introduce random variables (r.v.'s)

- r.v.: a real valued function $X(\omega)$ defined on Ω; $\forall x \in \mathbf{R}$ there defined the *(probability) cumulative distribution function (cdf)*

$$F(x) = \Pr\{X \leq x\}$$

- cdf $F(x)$: non-decreasing, $0 \leq F(x) \leq 1$, $F(-\infty) = 0$, $F(\infty) = 1$

Example (Throwing a fair dice)

can define X: the obtained number of the dice

$$X(\omega_1) = 1,\ X(\omega_2) = 2,\ X(\omega_3) = 3,\ X(\omega_4) = 4, \ldots$$

can also define X: indicator of whether the obtained number is even

$$X(\omega_1) = X(\omega_3) = X(\omega_5) = 0,\ X(\omega_2) = X(\omega_4) = X(\omega_6) = 1$$

Probability density and moments of distributions

- probability density function (pdf):

$$p(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

$$\Pr(a < X \le b) = \int_a^b p(x)\,\mathrm{d}x, \; a < b$$

sometimes we write $p_X(x)$ to emphasize that it is for the r.v. X

- mean, or expected value (first moment):

$$m_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x p_X(x)\,\mathrm{d}x$$

- variance (second moment):

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - m_X)^2\right] = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x)\,\mathrm{d}x$$

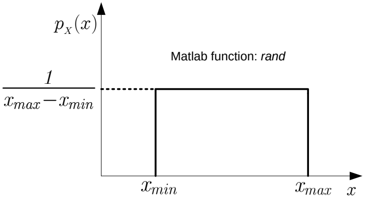
- standard deviation (std): $\sigma = \sqrt{\mathrm{Var}[X]}$
- exercise: prove that $\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2$

Example distributions

uniform distribution

- a r.v. uniformly distributed between x_{min} and x_{max}
- probability density function:

$$p(x) = \frac{1}{x_{\text{max}} - x_{\text{min}}}$$



- cumulative distribution function:

$$F(x) = \frac{x - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}}, \; x_{\text{min}} \le x \le x_{\text{max}}$$

- mean and variance:

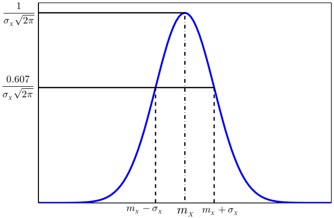
$$\text{E}[X] = \frac{1}{2}(x_{\text{max}} + x_{\text{min}}), \; \text{Var}[X] = \frac{(x_{\text{max}} - x_{\text{min}})^2}{12}$$

Example distributions

Gaussian/normal distribution

- importance: sum of independent r.v.s $\rightarrow$ a Gaussian distribution
- probability density function:

$$p(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{(x-m_X)^2}{2\sigma_X^2}\right)$$



- pdf fully characterized by m_X and σ_X. Hence a normal distribution is usually denoted as $N(m_X, \sigma_X)$
- nice properties: if X is Gaussian and Y is a linear function of X, then Y is Gaussian

Example distributions

Gaussian/normal distribution

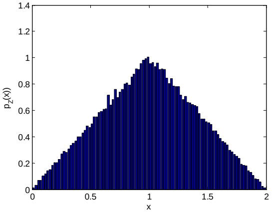
Central Limit Theorem: if X_1, X_2, ... are independent identically distributed random variables with mean m_X and variance σ_X^2, then

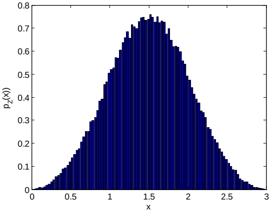
$$Z_n = \frac{\sum_{k=1}^{n} (X_k - m_X)}{\sqrt{n}\sigma_X^2}$$

converges in distribution to a normal random variable $X \sim N(0,1)$

example: sum of uniformly distributed random variables in $[0,1]$

```
X1 = rand(1,1e5);
X2 = rand(1,1e5);
X3 = rand(1,1e5);
Z = X1 + X2;
[fz,x] = hist(Z,100);
w_fz = x(end)/length(fz);
fz = fz/sum(fz)/w_fz;
figure, bar(x,fz)
xlabel 'x'; ylabel 'p_Z(x))';
Y = X1 + X2 + X3;
% ...
```





Multiple random variables

joint probability

for the same sample space Ω, multiple r.v.'s can be defined

- joint probability: $\Pr(X = x, Y = y)$
- joint cdf:

$$F(x,y) = \Pr(X \le x, Y \le y)$$

- joint pdf: $p(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$
- covariance:

$$\begin{aligned}\mathrm{Cov}(X,Y) = \Sigma_{XY} &= \mathrm{E}[(X - m_X)(Y - m_Y)] = \mathrm{E}[XY] - \mathrm{E}[X]E[Y] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - m_X)(y - m_Y)p(x,y)\mathrm{d}x\mathrm{d}y\end{aligned}$$

- uncorrelated: $\Sigma_{XY} = 0$
- independent random variables satisfy:

$$F(x,y) = \Pr(X \le x, Y \le y) = \Pr(X \le x)\Pr(Y \le y) = F_X(x)F_Y(y)$$
$$p(x,y) = p_X(x)p_Y(y)$$

Multiple random variables

more about correlation

correlation coefficient:

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

X and Y are uncorrelated if $\rho(X, Y) = 0$

- independent$\Rightarrow$uncorrelated; uncorrelated$\nRightarrow$independent
- uncorrelated indicates $\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 0$, which is weaker than X and Y being independent

Example

X–uniformly distributed on $[-1, 1]$. Construct Y: if $X \le 0$ then $Y = -X$; if $X > 0$ then $Y = X$. X and Y are uncorrelated due to

- $\mathrm{E}[X] = 0$, $\mathrm{E}[Y] = \frac{1}{2}$
- $\mathrm{E}[XY] = 0$

however X and Y are clearly dependent

Multiple random variables

random vector

- vector of r.v.'s:

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}$$

- mean:

$$m_Z = \begin{bmatrix} m_X \\ m_Y \end{bmatrix}$$

- covariance matrix:

$$\Sigma = \mathrm{E}\left[(Z - m_Z)(Z - m_Z)^T\right] = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} (x - m_X)^2 & (x - m_X)(y - m_Y) \\ (y - m_Y)(x - m_X) & (y - m_Y)^2 \end{bmatrix} p(x, y)\,\mathrm{d}x\mathrm{d}y$$

Conditional distributions

- joint pdf to single pdf:

$$p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}y$$

- conditional pdf:

$$p_X(x|y_1) = p_X(x|Y = y_1) = \frac{p(x,y_1)}{p_Y(y_1)}$$

- conditional mean:

$$\mathrm{E}[X|y_1] = \int_{-\infty}^{\infty} x p_X(x|y_1)\,\mathrm{d}x$$

- note: independent$\Rightarrow p_X(x|y_1) = p_X(x)$
- properties of conditional mean:

$$\mathop{\mathrm{E}}_{y}[\mathrm{E}[X|y]] = \mathrm{E}[X]$$

Multiple random variables

Gaussian random vectors

Gaussian r.v. is particularly important and interesting as its pdf is mathematically sound

Special case: two independent Gaussian r.v. X_1 and X_2

$$p(x_1,x_2) = p_{X_1}(x_1)\,p_{X_2}(x_2) = \frac{1}{\sigma_{X_1}\sqrt{2\pi}} e^{-\left(x_1-m_{X_1}\right)^2/\left(2\sigma_{X_1}^2\right)} \frac{1}{\sigma_{X_2}\sqrt{2\pi}} e^{-\left(x_2-m_{X_2}\right)^2/\left(2\sigma_{X_2}^2\right)}$$

$$= \frac{1}{\sigma_{X_1}\sigma_{X_2}\left(\sqrt{2\pi}\right)^2} \exp\left\{-\frac{1}{2}\begin{bmatrix} x_1-m_{X_1} \\ x_2-m_{X_2}\end{bmatrix}^T \begin{bmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2\end{bmatrix}^{-1} \begin{bmatrix} x_1-m_{X_1} \\ x_2-m_{X_2}\end{bmatrix}\right\}$$

We can use the random vector notation: $X = [X_1, X_2]^T$

$$\Sigma = \begin{bmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2\end{bmatrix}$$

and write

$$p_X(x) = \frac{1}{\left(\sqrt{2\pi}\right)^2\sqrt{\det\Sigma}} \exp\left\{-\frac{1}{2}[x-m_X]^T\Sigma^{-1}[x-m_X]\right\}$$

General Gaussian random vectors

pdf for a n-dimensional jointly distributed Gaussian random vector X:

$$p_X(x) = \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}[x - m_X]^T \Sigma^{-1} [x - m_X]\right\} \quad (1)$$

joint pdf for 2 Gaussian random vectors X (n-dimensional) and Y (m-dimensional):

$$p(x, y) = \frac{1}{\left(\sqrt{2\pi}\right)^{n+m} \sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}\begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}\right\} \quad (2)$$

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}$$

where Σ_{XY} is the cross covariance (matrix) between X and Y

$$\Sigma_{XY} = \mathrm{E}\left[(X - m_X)(Y - m_Y)^T\right] = \mathrm{E}\left[(Y - m_Y)(X - m_X)^T\right]^T = \Sigma_{YX}^T$$

General Gaussian random vectors

conditional mean and covariance

important facts about conditional mean and covariance:

$$m_{X|y} = m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]$$
$$\Sigma_{X|y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

proof uses $p(x,y) = p(x|y)\,p(y)$, (1), and (2)

- getting $\det\Sigma$ and the inverse Σ^{-1}: do a transformation

$$\begin{bmatrix} I & -\Sigma_{XY}\Sigma_{YY}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\Sigma_{YY}^{-1}\Sigma_{YX} & I \end{bmatrix} = \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix} \quad (3)$$

hence

$$\det\Sigma = \det\Sigma_{YY}\det\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right) \quad (4)$$

General Gaussian random vectors

inverse of the covariance matrix

computing the inverse Σ^{-1}:
–(3) gives

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I & 0 \\ -\Sigma_{YY}^{-1}\Sigma_{YX} & I \end{bmatrix} \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix}^{-1} \begin{bmatrix} I & -\Sigma_{XY}\Sigma_{YY}^{-1} \\ 0 & I \end{bmatrix}$$

–hence in (2):

$$\begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} x - m_X \\ y - m_Y \end{bmatrix}$$

$$= \begin{bmatrix} \star \end{bmatrix}^T \begin{bmatrix} \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} & 0 \\ 0 & \Sigma_{YY} \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} x - \left(m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]\right) \\ y - m_Y \end{bmatrix}}_{[\star]} \tag{5}$$

General Gaussian random vectors

$p(x,y) = p(x|y)\,p(y) \Rightarrow p(x|y) = p(x,y)/p(y)$

- using (4) and (5) in (2), we get

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{\det \underbrace{\left(\Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right)}_{[\star\star]}}}$$

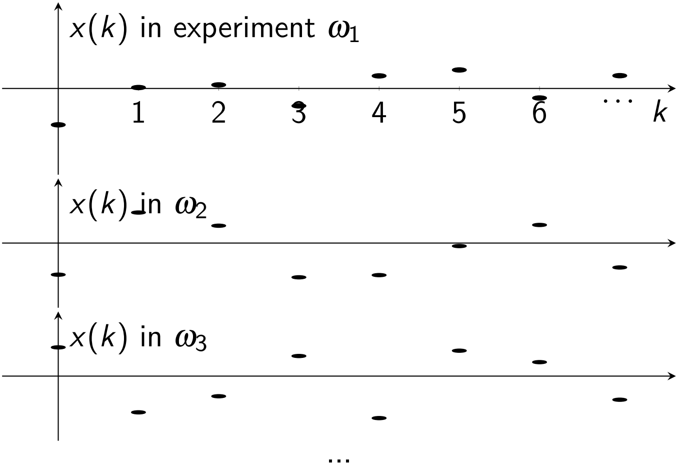
$$\times \exp\left\{-\frac{1}{2}\left[\underline{\ldots}\right]^T [\star\star]^{-1}\left[\underline{x - \left(m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]\right)}\right]\right\}$$

hence $X|y$ is also Gaussian, with

$$m_{X|y} = m_X + \Sigma_{XY}\Sigma_{YY}^{-1}[y - m_Y]$$
$$\Sigma_{X|y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

Random process

- discrete-time random process: a random variable evolving with time $\{x(k),\ k = 1, 2, \ldots\}$
- a stack of random vectors: $x(k) = [x(1), x(2), \ldots]^T$



Random process

$x(k) = [x(1), x(2), \ldots]^T$:

- complete probabilistic properties defined by the joint pdf $p(x(1), x(2), \ldots)$, which is usually difficult to get
- usually sufficient to know the mean $\mathrm{E}[x(k)] = m_x(k)$ and *auto-covariance*:

$$\mathrm{E}[(x(j) - m_x(j))(x(k) - m_x(k))] = \Sigma_{xx}(j,k) \tag{6}$$

- sometimes $\Sigma_{xx}(j,k)$ is also written as $X_{xx}(j,k)$

Random process

let $x(k)$ be a 1-d random process

- time average of $x(k)$:

$$\overline{x(k)} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{j=-N}^{N} x(j)$$

- ensemble average:

$$\mathrm{E}[x(k)] = m_{x(k)}$$

- *ergodic* random process: for all moments of the distribution, the ensemble averages equal the time averages

$$\mathrm{E}[x(k)] = \overline{x(k)},\ \Sigma_{xx}(j,k) = \overline{[x(j) - m_x][x(k) - m_x]}, \ldots$$

- ergodicity: not easy to test but many processes in practice are ergodic; extremely important as large samples can be expensive to collect in practice
- one necessary condition for ergodicity is stationarity

Random process

stationarity: tells whether the statistics characteristics changes w.r.t. time

- *stationary in the strict sense*: probability distribution does not change w.r.t. time

$$Pr\{x(k_1) \leq x_1, \ldots, x(k_n) \leq x_n\} = Pr\{x(k_1 + l) \leq x_1, \ldots, x(k_n + l) \leq x_n\}$$

- *stationary in the week/wide sense*: mean does not dependent on time

$$\mathrm{E}[x(k)] = m_x = \text{costant}$$

  and the auto-covariance (6) depends only on the time difference $l = j - k$

- can hence write

$$\mathrm{E}[(x(k) - m_x)(x(k+l) - m_x)] = \Sigma_{xx}(l) = X_{xx}(l)$$

- for stationary and ergodic random processes:

$$\Sigma_{xx}(l) = \mathrm{E}[(x(k) - m_x)(x(k+l) - m_x)] = \overline{(x(k) - m_x)(x(k+l) - m_x)}$$

Random process

covariance and correlation for stationary ergodic processes

- we will assume stationarity and ergodicity unless otherwise stated
- *auto-correlation*: $R_{xx}(l) = \mathrm{E}[x(k)x(k+l)]$.
- *cross-covariance*:

$$\Sigma_{xy}(l) = X_{xy}(l) = \mathrm{E}[(x(k) - m_x)(y(k+l) - m_y)]$$

- property (using ergodicity):

$$\begin{aligned}\Sigma_{xy}(l) = X_{xy}(l) &= \overline{(x(k) - m_x)(y(k+l) - m_y)} \\ &= \overline{(y(k+l) - m_y)(x(k) - m_x)} = X_{yx}(-l) = \Sigma_{yx}(-l)\end{aligned}$$

Random process

white noise

- *white* noise: a purely random process with $x(k)$ not correlated with $x(j)$ at all if $k \neq j$:

$$X_{xx}(0) = \sigma_{xx}^2,\ X_{xx}(l) = 0\ \forall l \neq 0$$

- non-stationary zero mean white noise:

$$\mathrm{E}[x(k)x(j)] = Q(k)\delta_{kj},\ \delta_{kj} = \begin{cases} 1 & ,\ k = j \\ 0 & ,\ k \neq j \end{cases}$$

Random process

auto-covariance and spectral density

- *spectral density*: the Fourier transform of auto-covariance

$$\Phi_{xx}(\omega) = \sum_{l=-\infty}^{\infty} X_{xx}(l)\, e^{-j\omega l},\ X_{xx}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{xx}(\omega)\,\mathrm{d}\omega$$

- *cross spectral density*:

$$\Phi_{xy}(\omega) = \sum_{l=-\infty}^{\infty} X_{xy}(l)\, e^{-j\omega l},\ X_{xy}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{xy}(\omega)\,\mathrm{d}\omega$$

properties:

- the variance of x is the area under the spectral density curve

$$\mathrm{Var}[x] = \mathrm{E}\left[(x - \mathrm{E}[x])^2\right] = X_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xy}(\omega)\,\mathrm{d}\omega$$

- $X_{xx}(0) \geq |X_{xx}(l)|,\ \forall l$

Filtering a random process

passing a random process $u(k)$ through an LTI system (convolution) generates another random process:

$$y(k) = g(k) * u(k) = \sum_{i=-\infty}^{\infty} g(i)\, u(k-i)$$

- if u is *zero mean* and ergodic, then

$$\begin{aligned} X_{uy}(l) &= \overline{u(k) \sum_{i=-\infty}^{\infty} u(k+l-i)\, g(i)} \\ &= \sum_{i=-\infty}^{\infty} \overline{u(k)\, u(k+l-i)}\, g(i) = \sum_{i=-\infty}^{\infty} X_{uu}(l-i)\, g(i) = g(l) * X_{uu}(l) \end{aligned}$$

similarly

$$X_{yy}(l) = \sum_{i=-\infty}^{\infty} X_{yu}(l-i)\, g(i) = g(l) * X_{yu}(l)$$

- in pictures:

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

Filtering a random process

input-output spectral density relation

for a general LTI system

$$u(k) \longrightarrow \boxed{G(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}} \longrightarrow y(k)$$

$$Y(z) = G(z)\,U(z) \Leftrightarrow Y\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) U\left(e^{j\omega}\right)$$

- auto-covariance relation in the last slide:

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

$$X_{yu}(l) = X_{uy}(-l) = g(-l) * X_{uu}(-l) = g(-l) * X_{uu}(l)$$

hence

$$\boxed{\Phi_{yy}(\omega) = G\left(e^{j\omega}\right) G\left(e^{-j\omega}\right) \Phi_{uu}(\omega) = \left|G\left(e^{j\omega}\right)\right|^2 \Phi_{uu}(\omega)}$$

Filtering a random process

MIMO case:

- if u and y are vectors, $G(z)$ becomes a transfer function matrix
- dimensions play important roles:

$$X_{uy}(l) = \mathrm{E}\left[(u(k) - m_u)(y(k+l) - m_y)^T\right] = X_{yu}(-l)^T$$

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l); \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

$$\begin{aligned} X_{yy}(l) &= g(l) * X_{yu}(l) = g(l) * X_{uy}^T(-l) \\ &= g(l) * [g(-l) * X_{uu}(-l)]^T \end{aligned}$$

$$\boxed{\Phi_{yy}\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) \cdot \Phi_{uu}\left(e^{j\omega}\right) G^T\left(e^{-j\omega}\right)}$$

Filtering a random process in state space

consider: $w(k)$–zero mean, white, $\mathrm{E}\left[w(k)w(k)^T\right]=W(k)$ and

$$x(k+1)=A(k)x(k)+B_w(k)w(k) \tag{7}$$

assume random initial state $x(0)$ (uncorrelated to $w(k)$):

$$\mathrm{E}[x(0)]=m_{x_0},\ \mathrm{E}\left[(x(0)-m_{x_0})(x(0)-m_{x_0})^T\right]=X_0$$

- mean of state vector $x(k)$:

$$\boxed{m_x(k+1)=A(k)m_x(k)},\ m_x(0)=m_{x_0} \tag{8}$$

- covariance $X(k)=X_{xx}(k,k)$: (7)-(8)$\Rightarrow$

$$\boxed{X(k+1)=A(k)X(k)A^T(k)+B_w(k)W(k)B_w^T(k)},\ X(0)=X_o$$

- intuition: covariance is a "second-moment" statistical property

Filtering a random process in state space

dynamics of the mean:

$$\boxed{m_x(k+1) = A(k)\, m_x(k)},\ m_x(0) = m_{x_0}$$

dynamics of the covariance:

$$\boxed{X(k+1) = A(k)\, X(k)\, A^T(k) + B_w(k)\, W(k)\, B_w^T(k)},\ X(0) = X_o$$

- (steady state) if $A(k) = A$ and is stable, $B_w(k) = B_w$, and $w(k)$ is stationary $W(k) = W$, then

$$m_x(k) \to 0,\ X(k) \to \text{a steady state } X_{ss}$$

$$X_{ss} = AX_{ss}A^T + B_w W B_w^T\text{: discrete-time Lyapunov Eq.} \qquad (9)$$

$$X_{ss}(l) = E\left[x(k)\, x^T(k+l)\right] = X_{ss}\left(A^T\right)^l$$

$$X_{ss}(-l) = X_{ss}(l)^T = A^l X_{ss}$$

Filtering a random process in state space

Example

first-order system

$$x(k+1) = ax(x) + \sqrt{1-a^2}w(k),\ \mathrm{E}[w(k)] = 0,\ \mathrm{E}[w(k)w(j)] = W\delta_{kj}$$

with $|a| < 1$ and $x(0)$ uncorrelated with $w(k)$.
steady-state variance equation (9) becomes

$$X_{ss} = a^2 X_{ss} + \left(1-a^2\right) W \Rightarrow X_{ss} = W$$

and

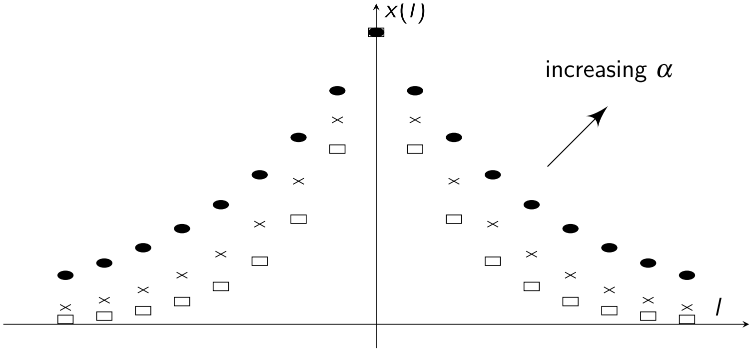
$$X(l) = X(-l) = a^l X_{ss} = a^l W$$

Filtering a random process in state space

Example

$$x(k+1) = ax(x) + \sqrt{1-a^2}w(k),\ \mathrm{E}[w(k)] = 0,\ \mathrm{E}[w(k)w(j)] = W\delta_{kj}$$

$$X(l) = X(-l) = a^l X_{ss} = a^l W$$



Filering a random process

continuous-time case

similar results hold in the continuous-time case:

$$u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

- spectral density (SISO case):

$$\Phi_{yy}(j\omega) = G(j\omega)\, G(-j\omega)\, \Phi_{uu}(j\omega) = |G(j\omega)|^2\, \Phi_{uu}(j\omega)$$

- mean and covariance dynamics:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + B_w w(t),\ \mathrm{E}[w(t)] = 0,\ \mathrm{Cov}[w(t)] = W$$

$$\frac{\mathrm{d}m_x(t)}{\mathrm{d}t} = Am_x(t),\ m_x(0) = m_{x_0}$$

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = AX + XA^T + B_w W B_w^T$$

- steady state: $X_{ss}(\tau) = X_{ss} e^{A^T \tau}$; $X_{ss}(-\tau) = e^{A\tau} X_{ss}$ where

$$AX_{ss} + X_{ss}A^T = -B_w W B_w^T : \text{ continuout-time Lyapunov Eq.}$$

Appendix: Lyapunov equations

- discrete-time case:

$$A^T PA - P = -Q$$

  has the following unique solution iff $\lambda_i(A)\lambda_j(A) \neq 1$ for all $i, j = 1, \ldots, n$:

$$P = \sum_{k=0}^{\infty} \left(A^T\right)^k QA^k$$

- continuous-time case:

$$A^T P + PA = -Q$$

  has the following unique solution iff $\lambda_i(A) + \bar{\lambda}_j(A) \neq 0$ for all $i, j = 1, \ldots, n$:

$$P = \int_0^{\infty} e^{A^T t} Q e^{At} \mathrm{d}t$$

Summary

1. Big picture

2. Basic concepts: sample space, events, probability axioms, random variable, pdf, cdf, probability distributions

3. Multiple random variables

   random vector, joint probability and distribution, conditional probability

   Gaussian case

4. Random process