LQ with Frequency Shaped Cost Function (FSLQ)

Background
Parseval's Theorem
Frequency-shaped LQ cost function
Transformation to a standard LQ

Big picture

why are we learning this:

- in standard LQ, Q and R are constant matrices in the cost function

$$J=\int_0^\infty \left(x^T(t)Qx(t)+\rho u^T(t)Ru(t)\right)\mathrm{d}t \tag{1}$$

- how can we introduce more design freedom for Q and R?

Connection between time and frequency domains

Theorem (Parseval's Theorem)

For a square integrable signal $f(t)$ defined on $[0, \infty)$

$$\int_0^\infty f^T(t) f(t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty F^T(-j\omega) F(j\omega) \mathrm{d}\omega$$

1D case:
$$\int_0^\infty |f(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty |F(j\omega)|^2 \mathrm{d}\omega$$

Intuition: energy in time-domain equals energy in frequency domain

For the general case, $f(t)$ can be acausal. We have

$$\int_{-\infty}^\infty f^T(t) f(t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty F^T(-j\omega) F(j\omega) \mathrm{d}\omega$$

Discrete-time version:

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \frac{1}{2\pi} \int_{-\infty}^\infty F^T\left(\mathrm{e}^{-j\omega}\right) F\left(\mathrm{e}^{j\omega}\right) \mathrm{d}\omega$$

History

Marc-Antoine Parseval (1755-1836):

- French mathematician
- published just five (but important) mathematical publications in total (source: Wikipedia.org)

Frequency-domain LQ cost function

From Parseval's Theorem, the LQ cost in frequency domain is

$$
\begin{aligned}
J &= \int_0^\infty \left(x^T(t)Qx(t) + \rho u^T(t)Ru(t)\right)\mathrm{d}t && (2)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(X^T(-j\omega)QX(j\omega) + \rho U^T(-j\omega)RU(j\omega)\right)\mathrm{d}\omega && (3)
\end{aligned}
$$

Frequency-shaped LQ expands Q and R to frequency-dependent functions:

$$
J = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(X^T(-j\omega)Q(j\omega)X(j\omega) + \rho U^T(-j\omega)R(j\omega)U(j\omega)\right)\mathrm{d}\omega \tag{4}
$$

Frequency-domain LQ cost function

Let

$$Q(j\omega) = Q_f^T(-j\omega)Q_f(j\omega) \succeq 0,\ X_f(j\omega) = Q_f(j\omega)X(j\omega)$$
$$R(j\omega) = R_f^T(-j\omega)R_f(j\omega) \succ 0,\ U_f(j\omega) = R_f(j\omega)U(j\omega)$$

(4) becomes

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(X_f^T(-j\omega)X_f(j\omega) + \rho U_f^T(-j\omega)U_f(j\omega)\right)\mathrm{d}\omega$$

which is equivalent to (using Parseval's Theorem again)

$$\boxed{J = \int_0^{\infty}\left(x_f^T(t)x_f(t) + \rho u_f^T(t)u_f(t)\right)\mathrm{d}t} \qquad (5)$$

Frequency-domain LQ cost function

Summarizing, we have:

- plant:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{6}$$

- new cost:

$$J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho\, u_f^T(t) u_f(t) \right) \mathrm{d}t \tag{7}$$

- filtered states and inputs:

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)\,, \quad u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

We just need to translate the problem to a standard one [which we know (very well) how to solve]

Frequency-domain weighting filters

state filtering

$$x(t) \longrightarrow \boxed{Q_f(s)} \longrightarrow x_f(t)$$

- a MIMO process in general: if $x(t) \in \mathbb{R}^n$ and $x_f(t) \in \mathbb{R}^q$, then $Q_f(s)$ is a $q \times n$ transfer function matrix
- $Q_f(s)$: state filter; designer's choice; can be selected to meet the desired control action and the performance requirements
- write $Q_f(s) = C_1(sI - A_1)^{-1} B_1 + D_1$ in the general state-space realization:

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 x(t) \\ x_f(t) = C_1 z_1(t) + D_1 x(t) \end{cases} \tag{8}$$

Frequency-domain weighting filters

input filtering

$$u(t) \longrightarrow \boxed{R_f(s)} \longrightarrow u_f(t)$$

- $R_f(s)$: input filter; designer's choice; can be selected to meet the robustness requirements
- write $R_f(s) = C_2(sI - A_2)^{-1} B_2 + D_2$ in the general state-space realization:

$$\begin{cases} \dot{z}_2(t) & = A_2 z_2(t) + B_2 u(t) \\ u_f(t) & = C_2 z_2(t) + D_2 u(t) \end{cases} \tag{9}$$

Back to time-domain design

Combining (6), (8) and (9) gives the enlarged system

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}}_{x_e(t)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(t)$$

and

$$x_f(t) = \underbrace{[D_1 \;\; C_1 \;\; 0]}_{C_e} \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}$$

$$u_f(t) = [0 \;\; 0 \;\; C_2] x_e(t) + D_2 u(t)$$

Summary of solution

With the enlarged system, the cost

$$J = \int_0^\infty \left(x_f^T(t) x_f(t) + \rho\, u_f^T(t) u_f(t) \right) \mathrm{d}t \tag{10}$$

translates to

$$J = \int_0^\infty \left(x_e^T(t) Q_e x_e(t) + 2u^T(t) \underbrace{\begin{bmatrix} 0 & 0 & \rho D_2^T C_2 \end{bmatrix}}_{N_e} x_e(t) + u^T(t) \underbrace{\rho D_2^T D_2}_{R_e} u(t) \right) \mathrm{d}t$$

$$Q_e = \begin{bmatrix} D_1^T D_1 & D_1^T C_1 & 0 \\ C_1^T D_1 & C_1^T C_1 & 0 \\ 0 & 0 & \rho C_2^T C_2 \end{bmatrix}$$

- solution (see appendix for more details):

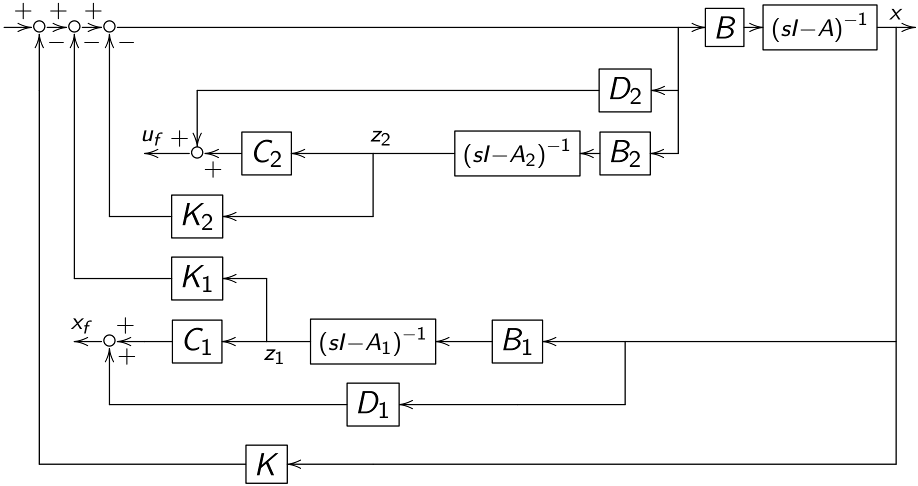
$$\boxed{u(t) = -R_e^{-1}(B_e^T P_e + N_e) x_e(t) = -Kx(t) - K_1 z_1(t) - K_2 z_2(t)}$$

- algebraic Riccati equation:

$$\boxed{A_e^T P_e + P_e A_e - (B_e^T P_e + N_e)^T R_e^{-1} (B_e^T P_e + N_e) + Q_e = 0}$$

Implementation

structure of the FSLQ system:



Appendix: general LQ solution

Consider LQ problems with cost

$$J = \int_0^\infty \left(x^T(t) \underbrace{C^T C}_{Q} x(t) + 2u^T(t) N x(t) + u^T(t) R u(t) \right) \mathrm{d}t \quad (11)$$

and system dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- assume (A, B) is controllable/stabilizable and (A, C) is observable/detectable
  - the solution of the problem is

$$u(t) = -R^{-1}(B^T P + N)x(t)$$

$$A^T P + PA - (B^T P + N)^T R^{-1}(B^T P + N) + Q = 0$$

Appendix: general LQ solution

Intuition: under the assumptions, we know we can stabilize the system and drive $x(t)$ to zero. Consider Lyapunov function $V(t) = x^T(t)Px(t)$, $P = P^T \succ 0$

$$
\begin{aligned}
\cancelto{0}{V(\infty)} - V(0) &= \int_0^\infty \dot{V}(t)\,\mathrm{d}t \\
&= \int_0^\infty \left(x^T(t)\left(PA + A^T P\right)x(t) + 2x^T(t)PBu(t)\right)\mathrm{d}t
\end{aligned}
$$

Adding (11) on both sides yields

$$
\begin{aligned}
&V(\infty) - V(0) + J = \\
&\int_0^\infty \left(x^T(t)\left(Q + PA + A^T P\right)x(t) + 2x^T(t)\left(PB + N^T\right)u(t) + u^T(t)Ru(t)\right)\mathrm{d}t
\end{aligned}
\tag{12}
$$

- to minimize the cost, we are going to re-organize the terms in (12) into some "squared" terms

Appendix: general LQ solution

"completing the squares":

$$2x^T(t)\left(PB+N^T\right)u(t)+u^T(t)Ru(t)=\left\|R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2$$
$$-x^T(t)\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)x(t)$$

hence (12) is actually

$$\cancelto{0}{V(\infty)}-V(0)+J$$
$$=\int_0^\infty\left[x^T(t)\left(Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)\right)x(t)\right.$$
$$\left.+\left\|R^{1/2}u(t)+R^{-1/2}\left(B^TP+N\right)x(t)\right\|_2^2\right]\mathrm{d}t$$

hence $J_{\min}=V(0)=x^T(0)Px(0)$ is achieved when

$$Q+PA+A^TP-\left(PB+N^T\right)R^{-1}\left(B^TP+N\right)=0$$

$$\text{and } u(t)=-R^{-1}\left(B^TP+N\right)x(t)$$