It follows that

$$\det(\gamma'(t), \gamma''(t), \gamma(t)) = -(\tau'(t))^3 \cdot \det(\gamma'(\tau(t)), \gamma''(\tau(t)), \gamma(\tau(t))).$$

Since both determinants are strictly positive, $\tau'(t) < 0$, i.e., τ is a strictly decreasing C^2-diffeomorphism of $\mathbb{R}$. Thus τ has a fixed point, i.e., there is t in $\mathbb{R}$ such that $\tau(t) = t$. But then $-\gamma(t) = \delta(t) = \gamma(\tau(t)) = \gamma(t)$, which is a contradiction. ∎

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Institut de Recherche Mathématique de Rennes, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France
huisman@univ-rennes1.fr
Home page: http://www.maths.univ-rennes1.fr/~huisman/

A Tile with Surround Number 2

Casey Mann

Consider the standard tiling of the Euclidean plane by unit squares: you see that surrounding any one square are eight squares. A different arrangement of unit squares can be found in which a square is completely surrounded by six squares. Similarly, a unit equilateral triangle can be completely surrounded by six unit equilateral triangles. A rectangle that is not a square can be completely surrounded by four copies of itself. In Figure 1 we show these examples. In [**1**], Friedman produced an example of a figure that can be completely surrounded by three copies of itself. An example based on Friedman's figure is given in Figure 2.

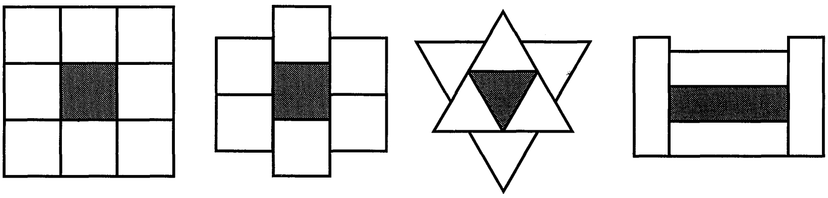

Figure 1. Basic examples.

This raises a natural question: Is there a figure that can be completely surrounded by just two copies of itself? We will show that the answer is yes. Moreover, we will go on to show that for each positive integer n there exists a figure such that n copies of this figure can be completely surrounded by only two copies of the figure.

First, we recall a few definitions. By *tile* we mean any figure in the Euclidean plane that is a closed topological disk. This includes polygonal figures and curvilinear figures such as lunes, but excludes figures with holes, figures with fractal boundaries,

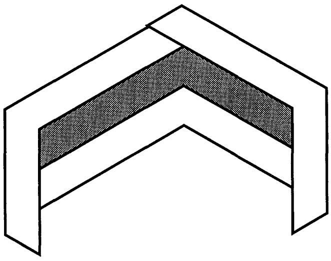

Figure 2. A figure surrounded by three copies of itself.

unbounded figures, and other bizarre shapes. Let T be a tile. A finite collection of copies of T is said to *surround* T if the interiors of the tiles in this collection along with T are pairwise disjoint and if their union contains all points within some fixed positive distance from T. For example, in Figures 1 and 2, each darkened shape is surrounded by the outer shapes, but in Figure 3 the darkened shapes are not surrounded. The minimum number of copies of T needed to surround T is called the *surround number* of T. In this language, we will present a tile with surround number 2.

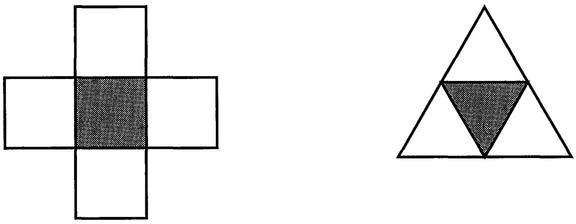

Figure 3. Shapes that are not surrounded.

The example we present here is based on the Voderberg tile [**6**], discovered in 1936 (Figure 4). The Voderberg tile, along with an infinite family of tiles that we'll call *generalized Voderberg tiles*, can be constructed using the method of Goldberg [**2**]. This construction is given in Figure 5.

The Voderberg tile is rather remarkable: along with other nice properties, it has the property that two copies can *almost* completely surround a third copy (Figure 4).

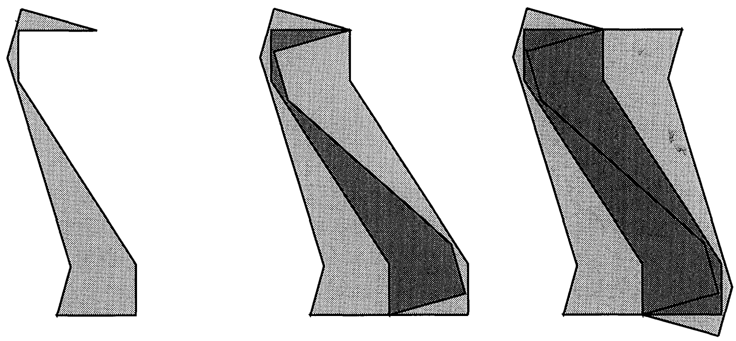

Figure 4. At the left is the Voderberg tile. It can *almost* be surrounded by two copies (center figure), and also two copies can *almost* be surrounded by two copies (right figure).

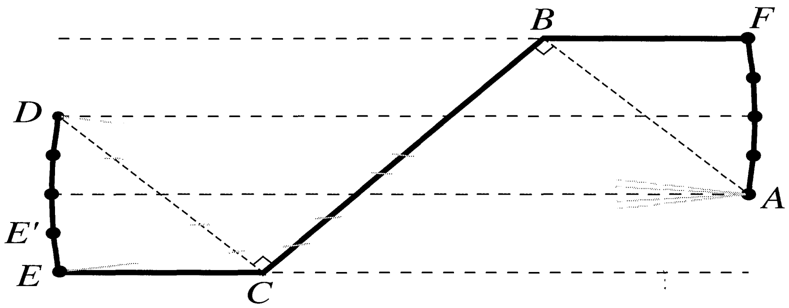

Figure 5. The construction of the generalized Voderberg tile. First construct four equidistant parallel lines, then make $ABCD$ with right angles at B and C. Let n be an even number ($n = 4$ here) and construct n equal line segments with endpoints on the circular arcs ED and AF, where A and D, respectively, are the centers of rotation. Let E' be the closest of the segment endpoints on arc ED to E. Let $S = AFBCE$. Let σ be the rotation about A taking E' to E. Then the figure bounded by S, $\sigma(S)$, and $E\sigma(E)$ is the generalized Voderberg tile that can *almost* surround $2n$ and $2n + 1$ copies of itself with only two copies.

In fact, the Voderberg tile has the property that two copies can *almost* completely surround two copies (Figure 4). Moreover, for every positive integer n there is a generalized Voderberg tile T such that two copies of T can almost completely surround n copies of T. An example of a generalized Voderberg tile in which two copies almost completely surround four copies is shown in Figure 6. As can be seen from these examples, there are always two exposed points that are not surrounded. What we aim to do in this paper is to make modifications to the Voderberg tiles that will correct this.

The modifications made to the Voderberg tile are the strategic placement of some "hooks" and "catches" (Figure 7). With these hooks and catches in place, it is clear how to fit the tiles together, and one sees that two copies of this tile now surround a third copy (Figure 8). With these alterations, this tile does not give rise to any tilings of the plane.

In the right-hand configuration of Figure 4, we see that two copies of the Voderberg tile can *almost* surround two copies. As an exercise, we suggest that the reader try to modify the Voderberg tile so that two copies completely surround two copies. (*Hint*: start by adding a small "tongue" to the rightmost copy to cover the exposed vertex at the top, then follow the implications around.) As we stated, the Voderberg tile can be generalized so that two copies of the tile can surround *any number* of copies of the tile, except for two points. By modifying these generalized Voderberg tiles in a manner

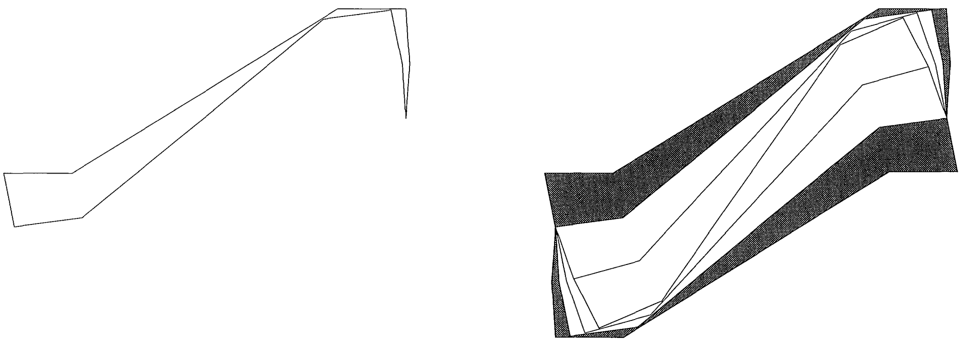

Figure 6. A generalized Voderberg tile is at left. At right, two copies *almost* surround four copies.

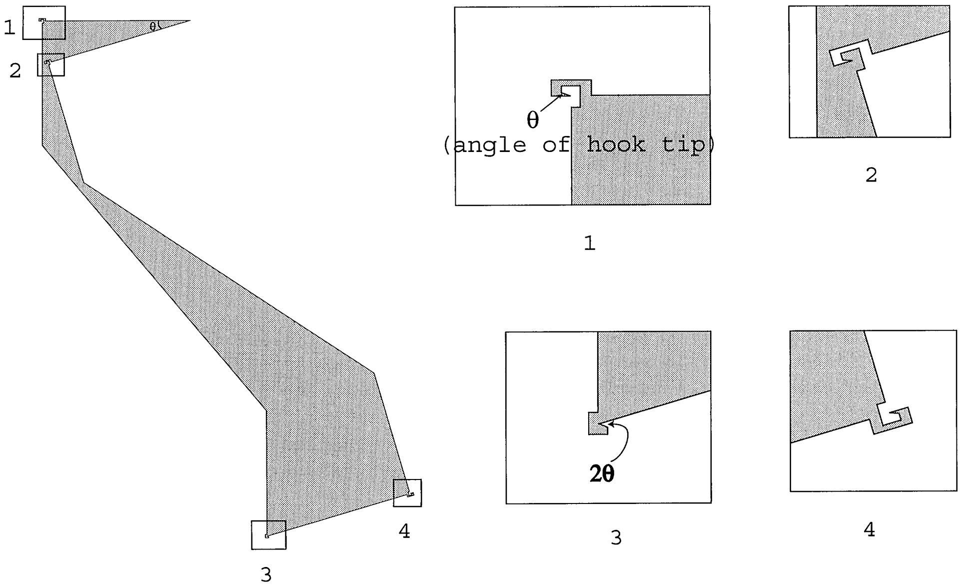

Figure 7. The modified Voderberg tile.

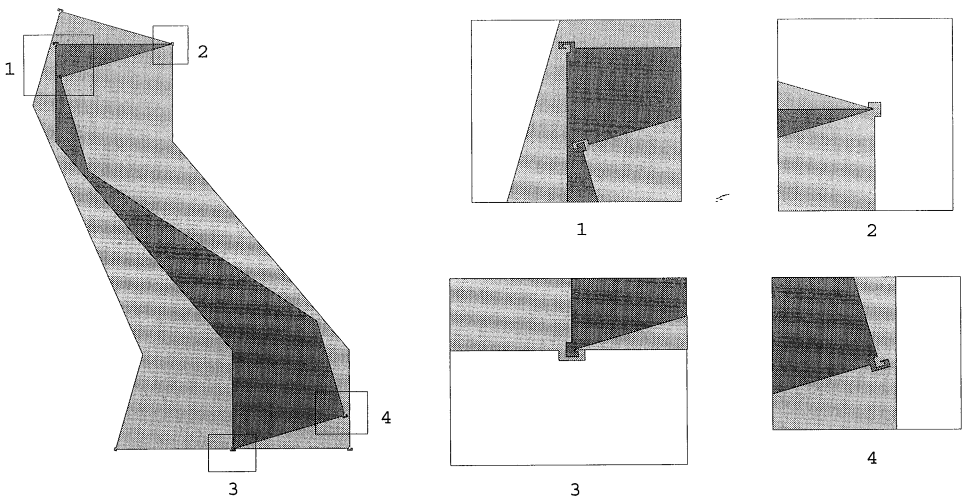

Figure 8. The modified Voderberg tile surrounding itself with two copies.

similar to what we did earlier, we can show that there are two distinct families of figures (one for surrounding even numbers of shapes, one for surrounding odd numbers of shapes) that together satisfy the following theorem:

Theorem 1. *For each integer $N \geq 1$, there exists a tile T_N for which two congruent copies of T_N can completely surround N copies of T_N.*

Our surround number 2 example disproves a conjecture made by Friedman in [**1**]. Friedman conjectured that no tile with surround number 2 (and Heesch number[1] 1)

[1]Roughly speaking, the *Heesch number* of a tile T is the maximum number of layers (i.e., rings or annuli) of copies of T that may surround T. Our example is easily seen to have Heesch number 1.

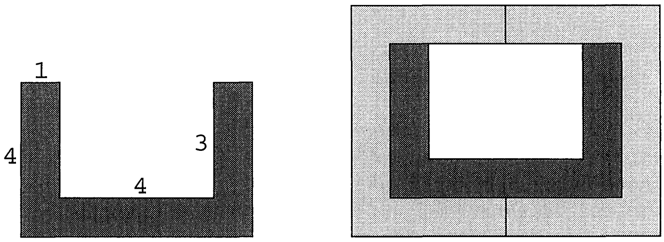

Figure 9. U-shaped polyomino that can form an annulus around itself with two copies, but a gap remains.

can exist. Indeed, this conjecture was reasonable; to see this, all one has to do is try to find such an example for oneself. It is not too hard to find tiles that *almost* have surround number 2. For example, one can easily find examples in which two copies of the tile form an annulus around a third, but leave unfilled gaps in the interior of the configuration, as in Figure 9. A three-dimensional analog of Figure 9 exists as well: namely, an open-top 6 by 8 by 10 box with walls of thickness 1, where the open top is on the 8 by 10 side. A few more related questions are posed by Grunbaum in [**4**] and in [**5**]: (1) Does there exist a tile T such that T has surround number 2 and T can be used to tile the entire Euclidean plane? (2) Does there exist a tile T that gives rise to a tiling of the plane in which a copy of T is surrounded by two copies of T? So far as this author knows, both of these problems are unsolved. Our example does not settle either question, for it cannot be used to tile the plane.

It is natural to ask about surround numbers of three-dimensional tiles. We do not know of any three-dimensional tiles with surround number 2 or 3; however, we can use the surround number 2 tile to generate a three-dimensional tile with surround number 4. This tile is a cylinder with the surround number 2 tile as its base. Here are the directions for its construction: make the height of this cylinder greater than the diameter of the base (of the cylinder); start with three cylinders whose bases are as in the left-hand configuration in Figure 8; finally, use the wide facet of a tile to cap the base and another to cap the top. This gives a tile with surround number 4 in three dimensions. In exactly this same way we can construct cylinders with the modified generalized Voderberg tiles as the bases to get three-dimensional tiles satisfying the following theorem:

Theorem 2. *For each integer $N \geq 1$, there exists a three-dimensional tile T_N for which four congruent copies of T_N can completely surround N copies of T_N.*

As we noted before, the open-top 6 by 8 by 10 box with walls of thickness 1 has the property that two copies form a shell about a third, but a gap remains. With this example in mind, we can see that if three-dimensional tiles with surround number 2 do not exist, then the reason might be very subtle. With that, we close with the following open question:

Open Question 1. *Do there exist three-dimensional tiles with surround numbers 2 and 3?*

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Northeastern State University, Tahlequah, OK 74464
mannce@nsuok.edu

On Stirling' s Formula

Reinhard Michel

1. INTRODUCTION. By means of simple but skillful estimates for the *log*-function we give a short, direct, and elementary proof of Stirling's formula and derive bounds for two asymptotic expansions.

2. STIRLING'S FORMULA. Well-known is $n! = \int_0^\infty x^n e^{-x}\,dx$. The substitution $x = y\sqrt{n} + n$ gives

$$n! = n^n \sqrt{n}\, e^{-n} \int_{-\infty}^{\infty} g_n(y)\,dy, \qquad n \in \mathbb{N}, \tag{1}$$

where $g_n(y) = (1 + \frac{y}{\sqrt{n}})^n e^{-y\sqrt{n}} 1_{(-\sqrt{n},\infty)}(y),\ y \in \mathbb{R}$. Since

$$\left|\log(1+x) - x + \frac{1}{2}x^2\right| \le \sum_{k=3}^{\infty} \frac{|x|^k}{k} \le \frac{1}{3}\frac{|x|^3}{1-|x|}, \qquad |x| < 1,$$

we obtain from $|e^a - e^b| = e^b|e^{a-b} - 1| \le e^b|a-b|e^{|a-b|}$ that

$$|g_n(y) - e^{-y^2/2}| \le e^{-y^2/2}\frac{2}{3}\frac{|y|^3}{\sqrt{n}}e^{y^2/3} \le \frac{|y|^3}{\sqrt{n}}e^{-y^2/6}, \qquad |y| \le \frac{1}{2}\sqrt{n}. \tag{2}$$

Note that (2) yields $\lim_{n\in\mathbb{N}} g_n(y) = e^{-y^2/2},\ y \in \mathbb{R}$.

Next, consider $f(x) = x - \frac{5}{6}\frac{x^2}{2+x} - \log(1+x),\ x > -1$. As $f'(x) = x\frac{x^2-x+4}{6(1+x)(2+x)^2}$, this function has an absolute minimum at $x = 0$. Therefore, $\log g_n(y) \le -\frac{5}{6}\frac{y^2}{2+y/\sqrt{n}}$, $y > -\sqrt{n}$, which implies that

$$0 \le g_n(y) \le e^{-|y|/6}, \qquad |y| > \frac{1}{2}\sqrt{n}. \tag{3}$$