Heteroskedasticity in Panel Data

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Review of heteroskedasticity

Recall that in cross-sectional LS, heteroskedasticity

- is assumed away
- if present, biases our standard errors

We noted two approaches

- Model the heteroskedasticity directly with an appropriate ML model, or
- Less optimally, continue to use the wrong method (LS), but try to correct the se’s; these are known as Huber-White, sandwich, or robust standard errors

How do these approaches transfer to the time series context? to panel data?
Dynamic heteroskedasticity

As with cross-sectional models, we can model heteroskedasticity directly.

One possibility is to let heteroskedasticity evolve dynamically.

We can let heteroskedasticity be (sort-of) “ARMA”, under the name “GARCH”.

Generalized Autoregressive Conditional Heteroskedasticity:

\[
\begin{align*}
y_t &= \mu_t + \varepsilon_t \\
\varepsilon_t &\sim f_N \left( 0, \sigma^2_t \right)
\end{align*}
\]

where

\[
\begin{align*}
\mu_t &= \alpha + x_t \beta + \sum_{p=1}^{P} y_{t-p} \phi_p + \sum_{q=1}^{Q} \varepsilon_{t-q} \theta_q \\
\sigma^2_t &= \exp \left( \eta + z_t \gamma \right) + \sum_{c=1}^{C} \sigma^2_{t-c} \lambda_c + \sum_{d=1}^{D} \varepsilon^2_{t-d} \xi_d
\end{align*}
\]

In words, \(y_t\) is an ARMA\((P, Q)\)-GARCH\((C, D)\) distributed time-series.

(Of course, we could leave out \(x\) and/or \(z\) if we wanted.)
**Dynamic heteroskedasticity**

\[ y_t = \mu_t + \varepsilon_t \quad \varepsilon_t \sim f_N (0, \sigma_t^2) \]

where

\[ \mu_t = \alpha + x_t \beta + \sum_{p=1}^{P} y_{t-p} \phi_p + \sum_{q=1}^{Q} \varepsilon_{t-q} \theta_q \]

\[ \sigma_t^2 = \exp (\eta + z_t \gamma) + \sum_{c=1}^{C} \sigma_{t-c}^2 \lambda_c + \sum_{d=1}^{D} \varepsilon_{t-d}^2 \xi_d \]

Models like the above are workhorses of financial forecasting

Can estimated by ML as usual

In \texttt{R}, \texttt{garch()} in the \texttt{tseries} package does GARCH

May have to look around a bit for ARMA-GARCH
Dynamic and panel heteroskedasticity

Panel data allows for more complex forms of heteroskedasticity and serial correlation than cross-sectional data. For example:

- **Serial correlation:** \( \mathbb{E}(\varepsilon_{is}\varepsilon_{it}) = \sigma_{st} \neq 0 \)

  (Reduced/eliminated by appropriate ARMA specification)

- **Contemporaneous correlation:** \( \mathbb{E}(\varepsilon_{it}\varepsilon_{jt}) = \sigma_{ij} \neq 0 \)

  (Globally reduced by year fixed effects, but what about pairs of correlated units?)

- **Panel heteroskedasticity:** \( \mathbb{E}(\varepsilon_{is}^2) = \mathbb{E}(\varepsilon_{it}^2) = \sigma_i^2, \text{ but } \sigma_i^2 \neq \sigma_j^2 \)

  (How to fix?)

- **Dynamic heteroskedasticity:** \( \sigma_{it}^2 = f(\sigma_{i,t-k}^2) \)

  (How to fix?)
Dynamic and panel heteroskedasticity

Consider a Panel ARMA-GARCH model:

\[ y_{it} = \mu_{it} + \varepsilon_{it} \]
\[ \varepsilon_{it} \sim f_{\mathcal{N}}(0, \sigma_{it}^2) \]

\[ \mu_{it} = \alpha_i + \tau_t + x_{it}\beta + \sum_{p=1}^{P} y_{i,t-p}\phi_p + \sum_{q=1}^{Q} \varepsilon_{i,t-q}\theta_q \]

\[ \sigma_{it}^2 = \exp(\eta_i + \zeta_t + z_{it}\gamma) + \sum_{c=1}^{C} \sigma_{i,t-c}^2\lambda_c + \sum_{d=1}^{D} \varepsilon_{i,t-d}^2\xi_d \]

Suppose we estimated \( y_{it} \) as a function of \( \mu_{it} \),
but ignored the structure of the error term

That is, we estimate a panel ARMA or GMM,
but assume \( \varepsilon_{it} \) is homoskedastic and serially uncorrelated, conditional on the covariates and lags in \( \mu_{it} \)
Dynamic and panel heteroskedasticity

If we ignore this:

$$\sigma_{i,t}^2 = \exp(\eta_i + \zeta_t + \omega_{it}) + \sum_{c=1}^{C} \sigma_{i,t-c}^2 \lambda_c + \sum_{d=1}^{D} \varepsilon_{i,t-d}^2 \xi_d$$

We would miss three non-standard features of the error variance-covariance:

- Panel heteroskedasticity, from $\eta_i$, the unit random effect in the variance function
- Contemporaneous correlation, from $\zeta_t$, the time random effect in the variance function
- Conditional heteroskedasticity: $\lambda_i$ and $\xi_i$ make the variance time dependent

Thankfully, few reasonable models are this complex...
Dynamic and panel heteroskedasticity

Suppose we think this AR(1) with panel heteroskedasticity is appropriate:

\[ y_{it} = \mu_{it} + \varepsilon_{it} \quad \varepsilon_{it} \sim f_N \left( 0, \sigma_i^2 \right) \]

\[ \mu_{it} = \alpha_i + x_{it} \beta + y_{i,t-p} \phi \]

\[ \sigma_i^2 = \exp(\eta_i) \]

Only source of heteroskedasticity is now \( \eta_i \):
panel heteroskedasticity, not dynamic heteroskedasticity

We could switch this to contemporaneous correlation, by swapping \( \zeta_t \) for \( \eta_i \)

Roughly the model Beck & Katz advocate as a baseline for comparative politics

Suggest estimating by LS then correcting se’s for omission of \( \eta_i \) & contemp. corr.

This procedure yields “panel-corrected standard errors”, PCSEs

What are they, and how do we compute them?
What would we do if we had a plain-vanilla cross-sectional regression and suspected or detected heteroskedasticity?

Recall the standard errors from LS are the square roots of the diagonal elements of

$$\hat{V}(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$$

So if $\sigma^2$ varies by $i$, these will be badly estimated

Instead, of the usual $\sigma^2$ estimator, we could use the residual from each observation as a robust estimate of its variance:

$$\hat{\sigma}_i^2 = \hat{\varepsilon}_i^2$$

A “heteroskedasticity robust” formula for the Var-Cov matrix follows:

$$\hat{V}(\hat{\beta}) = (X'X)^{-1}(\sum_i \hat{\varepsilon}_i x_i'x_i)(X'X)^{-1}$$

The standard errors of our parameters ($\beta$'s) are the square roots of the diagonal of this matrix
Review: Adjusting standard errors for heteroskedasticity

\[ \hat{V}(\hat{\beta}) = (X'X)^{-1} \left( \sum_i \hat{\varepsilon}_i x'_i x_i \right) (X'X)^{-1} \]

SE’s calculated from this equation are known by many names:

- Huber-White standard errors
- robust standard errors
- sandwich standard errors
- heteroskedasticity consistent standard errors

If you have a single time series, Newey-West standard errors generalize this concept to include robustness to serial correlation

For panel data there are many further options, leading to a vast literature exploring refinements to this basic concept
Panel-corrected standard errors

To calculate panel-corrected standard errors, we need to estimate the correct variance-covariance matrix.

Why not just use Huber-White? That would ignore panel structure, which is inefficient if we know how that structure affects heteroskedasticity:

\[ \hat{\sigma}_{it}^2 = \hat{\varepsilon}_{it}^2 \]

Beck and Katz’s panel correction produces sharper estimates of \( \hat{\sigma}_{it}^2 \) by borrowing strength across the observations from a single unit:

\[ \hat{\sigma}_{it}^2 = \hat{\sigma}_i^2 = \frac{1}{T}(\hat{\varepsilon}_{i,1}^2 + \hat{\varepsilon}_{i,2}^2 + \cdots + \hat{\varepsilon}_{i,T}^2) \]

Beck and Katz’s panel correction also accounts for contemporaneous correlations across units:

\[ \hat{\sigma}_{i,j} = \frac{1}{T}(\hat{\varepsilon}_{i,1}\hat{\varepsilon}_{j,1} + \hat{\varepsilon}_{i,2}\hat{\varepsilon}_{j,2} + \cdots + \hat{\varepsilon}_{i,T}\hat{\varepsilon}_{j,T}) \]

Note the above will work better if \( T \) is large relative to \( N \).
Panel-corrected standard errors

Building this intuition out into a variance-covariance matrix involves a bit of algebra.

To make PCSEs, suppose the variance-covariance matrix $\Omega$ is $NT \times NT$ block-diagonal with an $N \times N$ matrix $\Sigma$ of contemporaneous covariances on diagonal.

In other words, allow for unit or contemporaneous heteroskedasticity that stays the same over time.

Visualizing this large matrix is tricky.

Note that “$NT \times NT$ block-diagonal” means we are ordering the observations first by time, then by unit (reverse of our usual practice).
Panel-corrected standard errors

\[ \Omega_{NT \times NT} = \]

\[
\begin{bmatrix}
\sigma_1^2 & \sigma_1 \varepsilon_i & \sigma_1 \varepsilon_N & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\sigma_1 \varepsilon_i & \sigma_i^2 & \sigma_i \varepsilon_i & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\sigma_1 \varepsilon_N & \sigma_i \varepsilon_N & \sigma_i^2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i^2 & \sigma_i \varepsilon_i & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \sigma_i \varepsilon_N & \sigma_i \varepsilon_N & \cdots & \sigma_i^2 & \sigma_i \varepsilon_N & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i \\
0 & 0 & 0 & \sigma_i \varepsilon_N & \sigma_i \varepsilon_N & \cdots & \sigma_i \varepsilon_N & \cdots & \sigma_i \varepsilon_N & \cdots & \sigma_i \varepsilon_N & \cdots \\
0 & 0 & 0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \cdots \\
0 & 0 & 0 & 0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots & \sigma_i \varepsilon_i & \sigma_i \varepsilon_i & \cdots \\
\end{bmatrix}
\]
Panel-corrected standard errors

Instead, suppose $\Omega$ is $NT \times NT$ block-diagonal with an $N \times N$ matrix $\Sigma$ of contemporaneous covariances on diagonal.

In other words, allow for unit or contemporaneous heteroskedasticity that stays the same over time.

Beck and Katz (1995) estimate $\Sigma$ using LS residuals $e_{i,t}$:

$$\hat{\Sigma}_{i,j} = \sum_{t=1}^{T} \frac{e_{i,t}e_{j,t}}{T}$$

And then use $\hat{\Sigma}$ to construct the covariance matrix.
Panel-corrected standard errors

Monte Carlo experiments show panel-corrected standard errors are “correct” unless contemporaneous correlation is very high or $T$ is small relative to $N$.

(Note: alternative is to estimate random effects in variance by ML.)

Beck and Katz suggest using LS with PCSEs and lagged DVs as a baseline model.

Most practitioners think fixed effects should also be used.

Most important: getting the right lag structure & including FEs where appropriate.

PCSEs (or other var-cov correction) is a second-order concern.

In R, package pcse will calculate PCSEs for a linear regression.

Even easier: In the plm package, vcovBK() will produce a panel corrected var-cov matrix from a plm object.

If $N$ is large relative to $T$, consider the Driscoll and Kraay alternative, vcovSCC().
Panel-corrected standard errors: Application

Let’s apply Beck-Katz PCSEs to our panel ARIMA/plm example: we’ll replace the usual variance-covariance matrix with the panel corrected variance covariance matrix.

We must make this substitution manually after estimation to get corrected standard errors, confidence intervals, and var-cov matrices:

1. to print the summary() of a plm model

   Example:     summary(plm.res, .vcov=vcovBK(plm.res))

2. to use the coeftest() function on a plm model

   Example:     coeftest(plm.res, .vcov=vcovBK(plm.res))

3. to simulate parameters with mvrnorm() for computing counterfactuals

   Example:     mvrnorm(10000, coef(plm.res), vcovBK(plm.res))
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Recall the fixed effects results.

These are uncorrected for panel heteroskedasticity or contemporaneous correlation.
Panel correction usually makes little difference in long $T$ small $N$ contexts

But in short $T$, robust standard errors can be quite important. . .
Heteroskedastic and serial correlation consistent Var-Cov

In the PCSEs approach, the focus is on panel heteroskedasticity

It is assumed that serial correlation has been adequately modeled and purged

A reasonable check when we have a few dozen periods of data, though similar in most cases to either ordinary SEs or White SEs

But what if we have a low $T$? We might be more worried about residual serial correlation (and don't have practical access to ARMA diagnostics or fitting)

Now there is more need for a correction to the variance covariance that corrects for observed error correlation across units and across periods

Arellano (1987) provides a heteroskastic and autocorrelation consistent variance-covariance matrix: in plm, vcovHC()

Use the same commands as above, but with vcovHC() instead of vcovBK()

Particularly important to correct with panel GMM estimators
Our prior results for cigarette taxes used the Arellano heteroskedastic and serial correlation consistent var-cov matrix.

What would happen if we had used the ordinary, homoskedastic var-cov matrix to compute CIs?
The effects sizes are mostly unchanged: adjustments to standard errors affect CIs, not point estimates.

But the CIs are radically different under the traditional var-cov estimator.

Far too small (invisible even!) for the misspecified linear models.

And too large for the more correctly specified log-log models!
Just as panel GMM point estimate are sensitive to assumptions, so are the standard errors.

Use caution, and prefer \texttt{vcovHC()} to \texttt{vcov()} in PGMM models.

Be sure to check which var-cov matrix your functions are using; the default may be wrong!