Introduction to Bivariate Regression

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Motivating Example

We have cross-national data from several sources:

**Fertility**  The average number of children born per adult female, in 2000 (United Nations)

**Education Ratio**  The ratio of girls to boys in primary and secondary education, in 2000 (Word Bank Development Indicators)

**GDP per capita**  Economic activity in thousands of dollars, purchasing power parity in 2000 (Penn World Tables)

What are the levels of measurement of these variables?

Our question: how are these variables related to each other?
Motivating Example: Fertility, Female Education, and Development

Specifically, we ask:

1. If the level of female education changed by a certain amount, how much would we expect Fertility to change?
2. If the level of GDP per capita changed by a certain amount, how much would we expect Fertility to change?
3. How much would we expect our predictions to be off because of other random factors (noise)?
4. How much would we expect our predictions to be off because of sampling variability (poor estimation)?

Answering these questions will go far towards answering hypotheses about relationships between variables.
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Review the Univariate Summary Statistics for our Example

Explore the Bivariate Relationship between Fertility & Education Ratio

Explore the Bivariate Relationship between Fertility & GDP per capita

Throughout, develop a deeper understanding of linear regression
Summary of Univariate Distribution: Fertility

Median = 2.60
Mean = 3.12 children
std dev = 1.67 children
Summary of Univariate Distribution: Fertility

Median = 2.60
Mean = 3.12 children
std dev = 1.67 children

How would you describe this distribution?
Summary of Univariate Distribution: Education Ratio

Median = 99.60%
Mean = 94.48%
std. dev. = 12.45%
Summary of Univariate Distribution: Education Ratio

Median = 99.60%
Mean = 94.48%
std. dev. = 12.45%

How would you describe this distribution?
Summary of Univariate Distribution: GDP per capita

Median = $6047
Mean = $10,200
std. dev. = $10,078
Summary of Univariate Distribution: GDP per capita

Median = $6047
Mean = $10,200
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How would you describe this distribution?
How would you describe the relationship between Fertility & Education Ratio?

If I asked you to predict Fertility for a country not sampled, how accurate do you expect your prediction to be?
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If I asked you to predict Fertility for a country not sampled, how accurate do you expect your prediction to be?
Labelling cases sometimes helps, especially for identifying outliers.
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What makes a point an outlier?
The best fit line is the line that passes closest to the majority of the points.
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If we take this line to be our model of Fertility, how do we interpret it?
Best fit lines

Customarily, in statistics, we write the equation of a line as:

$$y = \beta_0 + \beta_1 x$$

where:

- $y_i$ is the dependent variable
- $x$ is the independent variable,
- $\beta_1$ is the slope of the line, or the change in $y$ for a 1 unit change in $x$,
- and $\beta_0$ is the intercept, or value of $y$ when $x = 0$
The above equation is the best fit line given by *linear regression*

The \( \hat{\beta} \)'s are the estimated linear regression *coefficients*

\( \hat{\text{Fertility}} = \hat{\beta}_0 + \hat{\beta}_1 \text{EduRatio} \)
\( \hat{\text{Fertility}} = 12.59 - 0.10 \times \text{EduRatio} \)

\( \hat{\text{Fertility}} \) is the *fitted value*, or model prediction, of the level of Fertility given the EduRatio
Intrepreting regression coefficients

\[
\hat{\text{Fertility}} = \hat{\beta}_0 + \hat{\beta}_1 \text{EduRatio}
\]

\[
\hat{\text{Fertility}} = 12.59 - 0.10 \times \text{EduRatio}
\]

Interpreting \( \hat{\beta}_1 = -0.10 \): 

*Increasing EduRatio by 1 unit lowers Fertility by 0.10 units.*

Because EduRatio is measured in percentage points, this means a 10% increase in female education (relative to males) will lower the number of children a woman has over her lifetime by 1 on average.
Intrepreting regression intercepts

\[
\begin{align*}
\hat{\text{Fertility}} &= \hat{\beta}_0 + \hat{\beta}_1 \text{EduRatio} \\
\hat{\text{Fertility}} &= 12.59 - 0.10 \text{EduRatio}
\end{align*}
\]

Interpreting \( \hat{\beta}_0 = 12.59 \):

*If EduRatio is 0, Fertility will be 12.59.*

If there are no girls in primary or secondary education, then women are expected to have 12.59 children on average over their lifetimes.

Can we trust this prediction?
Interpreting \( \hat{\beta}_0 = 12.59 \):

*If EduRatio is 0, Fertility will be 12.59.*

If there are no girls in primary or secondary education, then women are expected to have 12.59 children on average over their lifetimes.

Can we trust this prediction? No.

No country has 0 female education, so this is an *extrapolation* from the model.
Using regression coefficients to predict specific cases

\[ \hat{Fertility} = \hat{\beta}_0 + \hat{\beta}_1 \text{EduRatio} \]

\[ \hat{Fertility} = 12.59 - 0.10 \text{EduRatio} \]

How many children do we expect women to get if girls get half the education boys do?

*If EduRatio is 50, Fertility will be* \(12.59 - 0.10 \times 50 = 7.59\).

How many children do we expect women to have if girls get the same education boys do?

*If EduRatio is 100, Fertility will be* \(12.59 - 0.10 \times 100 = 2.59\).
Using regression coefficients to predict specific cases

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Does this hold exactly for any country with education parity?
Using regression coefficients to predict specific cases

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If EduRatio is 100, Fertility will be \(12.59 - 0.10 \times 100 = 2.59\).

Does this hold exactly for any country with education parity?

No. It holds on average. In any specific case \(i\), there is some error between the expected and actual levels of Fertility.
The linear regression model

\[ y_i = \beta_0 + \beta_1 x_i + \epsilon_i \]

To account for the random deviation of each case from the underlying trend, we add an *error term*, \( \epsilon_i \).

We will assume our \( y_i \)'s follow the above model.

That is, we will assume there is some “true” \( \beta_0 \) and \( \beta_1 \) which generated the \( y_i \) we observe, and some “true” error from this trend.
The linear regression model

\[ \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\epsilon}_i \]

When we estimate this model, we designate the estimates by adding “hats”

The estimates \((\hat{\beta}_0, \hat{\beta}_1, \hat{\epsilon}_i)\) probably differ from the (usually unknown) true values \((\beta_0, \beta_1, \epsilon_i)\)

To emphasize this, we will call \(\hat{\epsilon}_i\) the residual, since it is not the true error, but only an estimate
Estimating linear regression coefficients

\[ \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\epsilon}_i \]

How do we obtain our estimates of the $\beta$’s?

The full details are beyond the scope of 221

A key assumption is that $\epsilon_i$ is Normally distributed:

\[ \epsilon_i \sim \text{Normal}(0, \sigma^2) \]
The distribution of $\varepsilon_i$ determines how closely or widely the $y_i$’s are spaced around the best fit line.

Our key simplifying assumption is that everywhere around the line, the $y_i$’s are spread with the same Normal distribution.
With this assumption in mind, how do we find the best fit line?
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Perhaps the line that minimizes the total residuals?
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But some residuals are positive, and others negative—their sum is always 0
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Perhaps the line that minimizes the total residuals?

But some residuals are positive, and others negative—their sum is always 0.

So let's minimize the sum of squared error!

Linear regression is fitted using the *least squares* procedure.
The idea is to **minimize** the total spread of the y values from the line. Just as when we defined the variance, we look at all the **squared** y distances from the line, and add them up to get the **sum of squared errors**:

\[ SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \]

It's an aggregate measure of how much the line's "predicted \( y_i \)," or \( \hat{y}_i \), differ from the actual data values \( y_i \).

(Source: Larry Gonick & Wollcott Smith, *The Cartoon Guide to Statistics*)

The **least squares estimates** are the \( \hat{\beta} \)'s that minimize the total area of the above squares.
The idea is to minimize the total spread of the \( y \) values from the line. Just as when we defined the variance, we look at all the squared \( y \) distances from the line, and add them up to get the sum of squared errors:

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Statistics software can find these \( \hat{\beta} \)'s easily
Residuals

Notice the distinction between what we *explain* and what is left *unexplained*

\[ \hat{y}_i = a + bx_i \]

Thus, \( \hat{y}_i \) are the predicted weights determined by the regression line.

(Source: Larry Gonick & Wollcott Smith, *The Cartoon Guide to Statistics*)
Analysis of variance

The total variation in $y_i$ is its total variance from the mean $\bar{y}$, or $\sum_{i=1}^{n} (y_i - \bar{y})^2$

Using least squares, we can break down the variance in $y_i$ into two components:

- **Sum of square errors (SSE)** $\sum_{i=1}^{n} (y_i - \hat{y}_i)^2$
- **Regression sum of squares (RSS)** $\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$

The Regression sum of squares (RSS) is what we have explained. The Sum of squared errors (SSE) is what is left unexplained.
Analysis of variance

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- **Sum of square errors (SSE)** \(\sum_{i=1}^{n} (y_i - \hat{y}_i)^2\)
- **Regression sum of squares (RSS)** \(\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2\)
- **Total sum of squares (TSS)** \(\sum_{i=1}^{n} (y_i - \bar{y})^2\)

The Regression sum of squares (RSS) is what we have explained

The Sum of squared errors (SSE) is what is left unexplained
Analysis of variance

The Sum of squared errors is what is left unexplained:

\[ \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \]
Analysis of variance

The Sum of squared errors is what is left unexplained:

\[ \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \hat{e}_i^2 \]
Analysis of variance

The Sum of squared errors is what is left unexplained:

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \hat{\varepsilon}_i^2$$

A very useful summary of this is the square root of the mean squared error:

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2}$$
Analysis of variance

The Sum of squared errors is what is left unexplained:

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\[
\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2}
\]

This is how much a prediction from this linear regression will differ from the true \( y_i \) on average

Also known as the \textit{standard error of the regression}
The residuals for the regression of Fertility on Education Ratio

Chris Adolph (UW)
The residuals for the regression of Fertility on Education Ratio

This line minimizes the squared deviations on the dependent variable.
The smaller the sum of squared residuals, the better the model fits the data.
The smaller the sum of squared residuals, the better the model fits the data.

The quality of model fit is a separate issue from the substantive strength of the relationship, which is given by $\beta$, or the change in $y$ for a one unit change in $x$. 
Our model is captured in the $\beta$’s, or regression coefficients. In contrast to...

The correlation coefficient $r$, a goodness of fit measure; larger values imply better fit of the model to the data.

In our example, $r$ between Fertility and Education Ratio is $-0.75$.

Substantively, this number is hard to interpret. (What’s a “big” $r$? A “small” $r$? Arbitrary)
The coefficient of determination, $R^2$

One easy to interpret goodness of fit measure is $R^2$, known as the coefficient of determination

In general, $R^2$ is the ratio of the variance the model explains to the total variance:

$$R^2 = \frac{\text{RSS}}{\text{TSS}} = 1 - \frac{\text{SSE}}{\text{TSS}}$$

In bivariate regression only, $R^2$ also the square of $r_{X,Y}$

In our example, $R^2 = 0.56$, which says that Education Ratio “explains” 56% of the variation in Fertility, and vice versa

$R^2$ is a proportional reduction in error (PRE) statistic
I prefer a more tangible measure of goodness of fit, the root mean squared error (RMSE).
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RMSE is “how much your model predictions miss by”: 
I prefer a more tangible measure of goodness of fit, the root mean squared error (RMSE).

RMSE is “how much your model predictions miss by”:

here, 1.12 children per female
RMSE is better than $R^2$ because it can be compared across models and datasets—$R^2$ can’t.
Female Students as % of Male

Fertility Rate

RMSE is better than $R^2$ because it can be compared across models and datasets—$R^2$ can’t.

A question: we assumed the errors would be Normal—are they?
Recall that linear regression assumes the $\varepsilon_i$’s are Normally distributed.

![Graph showing residuals from Fertility vs Edu Ratio](image)
Recall that linear regression assumes the \( \varepsilon_i \)'s are Normally distributed.

We do \textit{not} assume that \( y_i \) follows a bell curve, except after controlling for \( x_i \)
Recall that linear regression assumes the $\varepsilon_i$'s are Normally distributed.

We do not assume that $y_i$ follows a bell curve, except after controlling for $x_i$.

Do the residuals appear Normally distributed in this case?
Uncertainty of $\hat{\beta}$

When estimating a mean or difference of means, we worried that by chance, our sample might not reflect the population.

That’s a worry in linear regression as well.

Does $\hat{\beta}$ estimated from our sample reflect the true population $\beta$?

Or did we get an unusual result due to sampling variability?
Uncertainty of $\hat{\beta}$

As with estimating a mean, we can calculate the *standard error* of $\hat{\beta}$.
Uncertainty of $\hat{\beta}$

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$se(\hat{\beta})$ is the amount we expect to miss the population $\beta$ by on average over regression using repeated samples
Uncertainty of $\hat{\beta}$

As with estimating a mean, we can calculate the \textit{standard error of} $\hat{\beta}$

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Remarkably, the $\hat{\beta}$’s themselves are Normally distributed, no matter what $y_i$ we are modeling
Uncertainty of $\hat{\beta}$

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So we can use a $t$-test to see if our $\hat{\beta}$’s would differ from the null hypothesis purely by chance
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Often, we will consider the null hypothesis to be $\beta^{\text{null}} = 0$, but sometimes we might want a different null
Uncertainty of $\hat{\beta}$

We can also construct confidence intervals around $\hat{\beta}_0$ and $\hat{\beta}_1$

These CIs reflect the uncertainty created by randomly sampling our data from the population

In 95% of samples, the true population $\beta$’s should lie in their 95% confidence intervals

Roughly, these intervals will be $\pm 2$ standard errors, if we have a lot of data
The standard errors of $\hat{\beta}$ reflect the fact that in 95% of randomly sampled datasets, the true best fit line for the population lies within range of the estimated line.
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We can capture this "wiggle room" graphically.
Why don’t 95% of the datapoints lie inside this interval?
Why don’t 95% of the **datapoints** lie inside this interval?

Because of fundamental uncertainty, or RMSE
Why don’t 95% of the *datapoints* lie inside this interval?

Because of fundamental uncertainty, or RMSE

The CIs just measure uncertainty in the best fit line, not in the data itself.
A standard regression table

Regression of Fertility on Education Ratio

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimates</th>
<th>se</th>
<th>t-stat</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>12.59</td>
<td>(0.75)</td>
<td>16.75</td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>Education Ratio</td>
<td>-0.10</td>
<td>(0.01)</td>
<td>-12.71</td>
<td>&lt; 0.001</td>
</tr>
</tbody>
</table>

<p>| | | | | |</p>
<table>
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<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>130</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>1.12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The most common presentation of a linear regression is the above table.

Usually, graphics are more informative and easier to read, but older articles rely heavily on this tabular format.

Understanding these tables will be important for the final exam. Let’s take this one apart.
A standard regression table

Regression of Fertility on Education Ratio

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| N                   | 130       |
| R^2                 | 0.56      |
| RMSE                | 1.12      |

The middle of the table contains several important quantities regarding our independent variable(s):

- **Estimates**: the $\hat{\beta}$’s, or regression coefficients
- **se**: the standard errors of $\hat{\beta}$
- **t-stat**: the t-statistic for the regression coefficient, or $\hat{\beta}/se(\hat{\beta})$
- **p-value**: the probability of seeing such a large t-stat by chance
Regression of Fertility on GDP per capita

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimates</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>12.59</td>
<td>[11.11, 14.08]</td>
</tr>
<tr>
<td>Education Ratio</td>
<td>−0.10</td>
<td>[−0.12, −0.08]</td>
</tr>
<tr>
<td>$N$</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.36</td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>1.35</td>
<td></td>
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</table>

Just as will our other estimates, we can construct confidence intervals around our $\hat{\beta}$’s.

Our results show 95% confidence that a 1 unit (1%) increase in education of girls relative to boys lowers fertility by between 0.08 and 0.12 children per woman.

We would only expect the truth to lie outside this interval in 1 of 20 random samples.
Wait a minute!

When we considered the relationship of female education and fertility, we also hypothesized an effect of GDP per capita.

We suspected this might be an indirect effect, flowing through female education.

Can we use regression to check for an effect of GDP?
Let's regress Fertility on GDP per capita.
Let’s regress Fertility on GDP per capita

Does this scatterplot suggest a linear relationship?
Not really. Later, we’ll discuss solutions for curved relationships.
Not really. Later, we’ll discuss solutions for curved relationships.

For now, let’s proceed with the best linear fit.
This is the least squares fit (What does that mean?)
This is the least squares fit (What does that mean?)

How good does this fit look?
Can you imagine an alternative model that would reduce the sum of squared residuals further?
Can you imagine an alternative model that would reduce the sum of squared residuals further?

Perhaps a concave curve?
Do the residuals look Normally distributed?
Do the residuals look Normally distributed?

A strongly skewed distribution of errors is cause for concern. More next week.
How do we interpret this 95% confidence interval?

Why don't 95% of the points lie inside it?
How do we interpret this 95% confidence interval?

Why don’t 95% of the points lie inside it?
### Regression of Fertility on GDP per capita

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<tr>
<td>Intercept</td>
<td>4.13</td>
<td>(0.17)</td>
<td>24.57</td>
<td>&lt; 0.001</td>
</tr>
<tr>
<td>GDP per capita ($k)</td>
<td>−0.10</td>
<td>(0.01)</td>
<td>−8.44</td>
<td>&lt; 0.001</td>
</tr>
</tbody>
</table>

- $N = 130$
- $R^2 = 0.36$
- RMSE = 1.35

How do we interpret this table?
Another regression table

Regression of Fertility on GDP per capita

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How much do we expect Fertility to change when we increase GDP by $1000?

What would Fertility be if GDP were $1000? $10,000? $30,000? 4.03, 3.13, and 1.13, respectively.

What would Fertility be if GDP were 0? Do you trust this estimate? 4.13. No—this is an extrapolation.
Another regression table

Regression of Fertility on GDP per capita

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1. How much do we expect Fertility to change when we increase GDP by $1000? *decrease* by 0.1 children

2. What would Fertility be if GDP were $1000? $10,000? $30,000? 4.03, 3.13, and 1.13, respectively.

3. What would Fertility be if GDP were 0? Do you trust this estimate? 4.13. No—this is an extrapolation.
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- *$R^2*$: 0.36
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Suppose we drew another sample of countries. Would we expect to see a GDP different from zero in that case?
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2. Why?
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1. Suppose we drew another sample of countries. Would we expect to see a GDP different from zero in that case? Yes.

2. Why?
   The se is small relative to $\hat{\beta}$, so the true $\beta$ is probably far from 0.

3. How likely is it that we would see a $t$ statistic this large if $\beta = 0$?
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| N                         | 130       |
| R²                        | 0.36      |
| RMSE                      | 1.35      |

1. Suppose we drew another sample of countries. Would we expect to see a GDP different from zero in that case? Yes.

2. Why?
   The se is small relative to $\hat{\beta}$, so the true $\beta$ is probably far from 0.

3. How likely is it that we would see a $t$ statistic this large if $\beta = 0$? Very unlikely—less than 1 in 1000 samples.
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What do these confidence intervals mean?
What do these confidence intervals mean? In 95% of random samples, the true \( \beta \)'s will lie inside these intervals.
**Another regression table**

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1. How much of the variance in Fertility does this model explain? 36 percent

2. When using the model to predict fertility for a specific country, how much does it miss by on average?
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Foreshadowing

How do we reconcile our two sets of results?

Which model, if any, is right?

To solve this conundrum, we need *multiple regression*:
A method for regressing a dependent variable on several independent variables at once

Then, at last, we can say something about confounders

Fortunately, all of today’s concepts will carry over to multiple regression
### Important linear regression concepts

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<tr>
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<th>Formula</th>
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<tr>
<td>Regression coefficient</td>
<td>$\beta$</td>
</tr>
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<td>$\hat{\beta}$</td>
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<td>Standard error of est. of reg. coef.</td>
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<td>$\hat{y}_i$</td>
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<td>$\sum_{i=1}^{n} \varepsilon_i$</td>
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<td>Regression sum of squares (SSR)</td>
<td>$\sum_{i=1}^{n} \hat{y}_i - \bar{y}$</td>
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