Keynote paper from the Proceedings of 2nd International Conference On Research, Implementation And Education Of Mathematics And Sciences (2nd ICRIEMS), Yogyakarta, May 17-19, 2015. Ed. E. Retnowati, Yogyakarta State University, Yogyakarta, Indonesia, ISBN 978-979-96880-8-8

Developing expert mathematization of physics in the introductory course: an impedance mismatch

Suzanne White Brahmia
Rutgers, the State University of New Jersey
brahmia@physics.rutgers.edu

Physics is required for most STEM degrees largely to help develop the habits of mind that typifies its professional practice. As the most mathematically creative of the introductory sciences, physics is strongly dependent on students' conceptual understanding of the basic algebra used, not just knowing how to do problems. How we mathematize our subject in physics is an important part of developing expertise, and students' weakness in this domain has been linked to increased likelihood of failing the course. This paper presents research findings from an experiment conducted at Rutgers University, in which over 700 first-year engineering students were assessed in their physics and chemistry courses in domains of conceptual algebraic thinking, and mathematical attitudes and beliefs. While these students are well prepared by traditional standards to take a calculus-based physics course (average Math SAT score of 680, comparable to UCLA and UC Berkeley), the data reveal significant weaknesses in students' algebraic reasoning both in common everyday and physical science contexts. We compare results from data collected in a traditionally-taught chemistry course and two physics courses, one that is mostly traditionally taught and designed for students co-registered in calculus and one that is designed for students less-well prepared mathematically and is rich in IE (interactive engagement) practices focused on developing mathematical reasoning. We see that while students' attitudes and beliefs as measured by the CLASS (Colorado Learning Attitudes about Science Survey) become less expert-like for the students taking the traditional courses and more expert-like in the IE course, neither the IE nor the traditional courses make dramatic improvement in students' spontaneous and appropriate reasoning about linear relationships between abstract quantities.

Keywords: Mathematization, quantification, physics education, mathematics education, chemistry education

Introduction

The 21st Century workplace demands participants who are able to reason mathematically in a variety of contexts. Specifically, the workforce is expected to make mathematical sense of our rapidly changing and increasingly technological society. The question for educators preparing students for the 21st century workforce is not how well do our mathematically strongest students fare in calculus, but more importantly how effectively do all students reason algebraically in science and technology-based contexts?

Physical science courses provide an opportunity for students to develop their skills applying basic algebraic reasoning to real-world contexts beyond the topics that involve familiar quantities typically seen in math class word problems (shopping, games, etc.) It is not so much a question of whether or not students can perform algebraic manipulations, but moreover do they think algebraically about new kinds of quantities when they aren't prompted to calculate? The former involves two kinds of thinking. The first is epistemological in nature – do students consider mathematical sensemaking to be a valuable tool that helps them to make sense of the natural world, and do they use it spontaneously? The second is whether they actually reason conceptually in the context of physical science with the mathematics they've learned in school. Together these ways of thinking form what is known as *mathematization* of physical science.

Addressing the first question associated with mathematization, researchers have articulated and developed instruments for measuring some attitudes and beliefs regarding the use of mathematics in physical science (Adams et al., 2006; Elby, Frederiksen, Schwarz, & White, 2003; Halloun & Hestenes, 1998; Redish, Saul, & Steinberg, 1998). There is a growing body of evidence that these attitudes and beliefs actually become less expert-like as a result of having taken a college level introductory physical science course (Adams et al., 2006; Redish, et al., 1998).The CLASS (Colorado Learning Attitudes about Science Survey) (Adams et al., 2006) was built upon existing attitudes/beliefs surveys to focus on students' beliefs about physics, learning physics and problem solving in physics. Researchers have noted that in the rare cases when improved attitudes are observed they often accompany improved learning outcomes as measured by physics concept inventories (Brahmia, Boudreaux, & Kanim, 2015; Brewe, Traxler, de, Jorge, & Kramer, 2013; Lindsey, Hsu, Sadaghiani, Taylor, & Cummings, 2012; Otero & Gray, 2008)

The question of whether students think conceptually using the mathematics that is a prerequisite for the course they are taking has been getting some attention in recent years. Redish and Kuo (Redish & Kuo, 2015) state, "Despite much research on the learning of both physics and math, the problem of how to effectively include math in physics in a way that reaches most students remains unsolved." Rebello and collaborators (Rebello, Cui, Bennett, Zollman, & Ozimek, 2007) conducted a study that focused on measuring horizontal transfer between prerequisite math courses and subsequent physics courses, i.e. the spontaneous use of mathematics in physics that calls on math already learned in a prior math course. The researchers interviewed calculus-based physics students at Kansas State University, asking them to solve physics problems that involved simple integration or differentiation similar to physics problems they had already solved in homework. While the students were able to procedurally do the calculus required in a physics problem when it was prompted (as it typically is in many end-of-chapter problems) they were largely unsuccessful in the interviews at setting up and solving problems that required them to select appropriate calculus tools and adapt them to fit a physical situation. The researchers also surveyed algebra-based students before and after instruction using both a trigonometry survey and a set of trigonometry-based physics problems, and found little evidence of horizontal transfer of trigonometry from math to physics for these students. The authors report "…we had assumed that the problems would involve horizontal transfer and therefore be perceived as relatively straightforward by the students. It appears however that this was not the case with most students."

Although instructors commonly use calculus to derive equations as part of their explanations in physics and chemistry courses, the actual creative problem solving that students engage in, even in calculus-based courses, is largely algebraic in nature. And in a calculus-based course the instructors assume, as Rebello et. al did in their research, that students have a conceptual understanding of algebra before taking the course. The introductory courses in physics and chemistry introduce hundreds of new physical quantities that are largely unit rates or product quantities formed by combining newly learned quantities into even newer ones (eg molarity, density, acceleration, electric field). Proportional reasoning with these quantities (i.e., the reasoning about linear relationships in physics and chemistry that includes ratio reasoning, scaling and covariational reasoning) is ubiquitous as students learn to reason and make sense mathematically.

This paper describes a novel effort to begin addressing the questions: How well do freshmen students reason about linear relations before, during and after taking an introductory physical science course? And, does improving student attitudes and beliefs about the use of mathematics in science help facilitate students' spontaneous and appropriate algebraic thinking in science courses?

There currently are no validated concept inventories that probe deeply into this specific aspect of physics learning. Prior studies that involve proportional reasoning in physics have largely used the Lawson Classroom Test of Scientific Reasoning (Lawson, 1978) to correlate reasoning ability at entry into the introductory physics course with pre-post learning gains (e.g., as measured by the Force Concept Inventory). We join math education researchers (Thompson & Saldanha, 2003) in the view that proportional reasoning is multifaceted and generally context dependent. For our study we wanted a better assessment of student reasoning than the Lawson test, which was developed as a way of gauging the Piagetian developmental level of students. The Lawson test contains just four proportional reasoning items. We developed a suite of questions (reported on in Boudreaux, Kanim and Brahmia (Boudreaux, Kanim, & Brahmia, 2015)) that probe proportional reasoning as it is commonly used in the introductory course. This suite of questions was used in the study reported on here.

In addition, this paper describes changes in students' attitudes and beliefs about the use of mathematics in the context of their physics course, as measured by the CLASS in chemistry and in physics. The chemistry version of the CLASS, the CLASS-Chem, is largely similar to the CLASS-physics with some additional statements, such as those on the atomic-molecular perspective of chemistry. We report on the categories from the CLASS that are directly related to student attitudes about the use of mathematics in physical science.

We compare and contrast two populations. One population includes engineering students that are simultaneously enrolled in a large enrollment traditionally taught calculus-based physics course with minor IE (interactive engagement) modifications (clickers, short weekly minilab), a traditionally taught calculus-based chemistry course, and a traditionally taught math course at the level of introductory calculus or higher. The other population includes engineering students enrolled in a transformed calculus-based physics course (Brahmia, 2008) designed for students underprepared mathematically, about half of whom are also enrolled in the same calculus-based chemistry as the comparison group, and ~80% of whom are co-registered in a traditionally taught pre-calculus math course their first semester of study. We address the following research questions:

1. Are students enrolled in a typical sequence of traditionally taught, first-year engineering courses likely to become more effective algebraic thinkers after one and two semesters of instruction, and can the level of intervention targeted at developing mathematical reasoning impact these outcomes?
2. Are students enrolled in a typical sequence of traditionally taught, first-year engineering courses likely to adopt expert-like attitudes and beliefs regarding mathematization in chemistry and physics, and can the level of intervention targeted at developing mathematical reasoning impact these outcomes?

If as a research community we develop a more refined understanding of how students learn to mathematize in our courses, this knowledge can inform instructors and curriculum designers to help create learning environments that produce students who reason scientifically and are better prepared for either the continuation of studies in a STEM discipline, or for being a productive participant in the 21st century workplace regardless of their major field of study. This paper and its findings represent a step in that direction.

Research Methods

The first phase of this project involved the development of questions that probe students' spontaneous and appropriate proportional reasoning in novel contexts, and is reported on in Boudreaux et al. (Boudreaux, et al., 2015)

Question development

In collaboration between Andrew Boudreaux, Stephen Kanim and this paper's author a large number of written questions to gauge student facility with proportional reasoning were developed (Boudreaux, et al., 2015). Most of the questions were designed to focus on a single reasoning sub-skill of proportional reasoning. In all cases neither a calculator nor multistep problem solving are required.

The items underwent repeated cycles of validation and modification over a 3-year period. A sample of the questions is presented in Table II of this paper in their final form. Initially, all items asked students to explain their reasoning and show their work. We collected written responses from over 1000 students enrolled in a variety of courses (general education physics, introductory calculus-based and algebra-based physics, and physics for pre-service elementary teachers) taught at the researchers' home institutions. We created scoring rubrics based on student responses. Three coders initially scored tests to establish inter-rater reliability before we analyzed the data. We then created multiple-choice versions of the questions, with distractors based on the student difficulties identified during the analysis of written work and interviews. We established question validity through interviews conducted with individual students at WWU (Western Washington University). Results from the interviews and from the analysis of written responses were used to guide modifications to improve not only the clarity but also the effectiveness of the distractors in characterizing student reasoning.

Boudreaux and colleagues at WWU conducted the interviews with student volunteers from calculus-based introductory physics courses, general education physics courses, and a physics course for pre-service elementary teachers. They conducted over 20 interviews with each interview lasting about one hour. The interviews were videotaped for later transcription and analysis. They used a semi-structured protocol. The interviewer posed specific proportional reasoning questions and asked the interview subject to "think out loud." The interviewer clarified the question as needed, prompted the subject to explain his or her thinking after periods of sustained silence, and asked the subject to elaborate on statements that were brief or unclear. The interviewer did *not* offer hints or guiding questions.

Large-scale study

The results presented in this paper derive from a large-scale study involving the freshman and sophomore non-honors engineering students at a Rutgers University (N = 2,115 pretests and 1,784 post tests administered in all). The students in this study were comparatively well prepared mathematically; their mean mathematics SAT (2011/2012 test version) score was 680 (comparable to UCLA and UC Berkeley (College Board, 2015))

We administered a suite of multiple-choice proportional reasoning items as an ungraded in-class pretest during the first week of the introductory, calculus-based physics courses and the calculus-based general chemistry course in fall 2013. The testing conditions were the same in the physics courses for the post-test, which was administered 10 days before the end of the semester. There was a substantial drop in number of students taking post-test in Chemistry due to the fact that the test was administered online outside of class. (see Table I).

Table I: Assessment administration Fall 2013 Rutgers University (engineering students)

Subject	*PreTest*	*PostTest*	*#versions*
Physics - Mechanics	Supervised In class	Supervised In class	8
General Chemistry	Supervised In class	Unsupervised Online	6

The suite administered to the freshmen contained 13 items in all, distributed on two tests in two different subjects: Seven items were administered on the test for mechanics students (n_{pre}=759 and n_{post}=769), and six on the test for chemistry students (n_{pre}=692 and n_{post}=494). The majority of the students (n=613) students took both the mechanics and the chemistry course simultaneously. For all classes, these proportional reasoning items were bundled with a standardized concept inventory (the Force Concept Inventory(FCI) (Hestenes, Wells, & Swackhamer, 1992) in the physics course and the Chemical Concepts Inventory(CCI) (Journal of Chemical Education, 2015) in the chemistry course.) In a single sitting, students first completed the proportional reasoning suite, followed immediately by the concept inventory. The students were not constrained by time and were awarded credit for participation. The students took an identical posttest, administered under the same conditions in physics and online in chemistry (see Table I), at the end of the course. We report on the percentage of students who selected the correct answers and the p-values from repeated t-tests.

In order to test the effect of surface features on student reasoning, we administered matched versions of selected proportional reasoning items on different versions the suite (see Table I.) The suite was administered in the recitation section of the course, and within a given recitation the versions were assigned randomly. We administered identical versions of the suite on the pretest and the posttest to each student in the study. These features are described in detail in a separate paper (Brahmia, Kanim, & Boudreaux, 2015) but relevant findings will be discussed here also.

Table II: Sample questions from the physics (mechanics) and the chemistry suites.

Sample from the Physics Suite	**Sample from the Chemistry Suite**
P1. A bicycle is equipped with an odometer to measure how far it travels. A cyclist rides the bicycle up a mountain road. When the odometer reading increases by 8 miles, the cyclist gains *H* vertical feet of elevation. Find an expression for the number of miles the odometer reading increases for every vertical foot of elevation gain. a. $\sin^{-1}(H/8)$ b. $\sin^{-1}(8/H)$ c. $H/8$ d. $8/H$ e. none of these **P2.** Consider the following statement about Winnie the Pooh's dream: *"There are three times as many heffalumps as woozles."* Some students were asked to write an equation to represent this statement, using *h* for the number of heffalumps and *w* for the number of woozles. Which of the following is correct? a. $3h/w$ b. $3h = w$ c. $3h + w$ d. $h = 3w$ e. both a and b	**C1.** Catherine shuffles her feet across her living room carpet and then she touches a doorknob, which has a surface area of 30 square centimeters. When she touches the doorknob she transfers 3 microcoulombs of electric charge that spreads out uniformly over the doorknob's surface. Catherine divides 30 by 3 and gets 10. Which of the following statements about the number 10 is true? 10 is the total number of… a. microcoulombs of charge transferred b. square centimeters of surface area covered by the charge c. microcoulombs of charge that covers one square centimeter d. square centimeters that one microcoulomb of charge covers e. none of the above **C2.** You are part of a team that has invented a new, high-tech material called "traxolene." One gram of traxolene has a volume of 0.41 cubic centimeters. For a laboratory experiment, you are working with a piece of traxolene that has a volume of *N* cubic centimeters. Which of the following expressions helps figure out the mass of this piece of traxolene (in grams)? a. N/0.41 b. 0.41/N c. $N\cdot 0.41$ d. $(N + 1)\cdot 0.41$ e.none of these

In addition to the in-class testing, the students took the CLASS-physics and CLASS-chemistry online as a pretest during the first week of class, mid-year and as a post test during the last week of class. The CLASS is comprised of 42 statements in physics, and 50 in chemistry, with which experts clearly agree or disagree. Students rate the statements using a 5 point Likert scale, where 1 represents "completely disagree", 3 is neutral and 5 represents "completely agree". Example statements are:

> *"In physics(chemistry), it is important for me to make sense out of formulas before I can use them correctly."*
>
> *"If I want to apply a method used for solving one physics(chemistry) problem to another problem, the problems must involve very similar situations."*

Students are then scored based on the percentage of statements about which they are in agreement with experts, and on the percentage of statements about which they disagree. Students can neither agree nor disagree by selecting "neutral".

Completion of the pre- and post- online CLASS replaced a quiz grade or provided extra points on a test in all courses. Many students in the traditionally taught physics and chemistry courses either did not participate in the post-test, or did not agree to have their data be part of this study. We attribute this in part to survey fatigue. We report on matched pre/post results, and we have eliminated any tests that showed patterns of low effort (long strings of same answer).

Results

Proportional Reasoning Survey

Table III shows the matched set of pre, mid-year and post test scores on the suite of questions that each group was asked. The students in both mechanics courses received identical questions, and the chemistry students received a different set of questions that spanned a similar set of proportional reasoning constructs. Table II includes sample questions from both the mechanics and the chemistry suite.

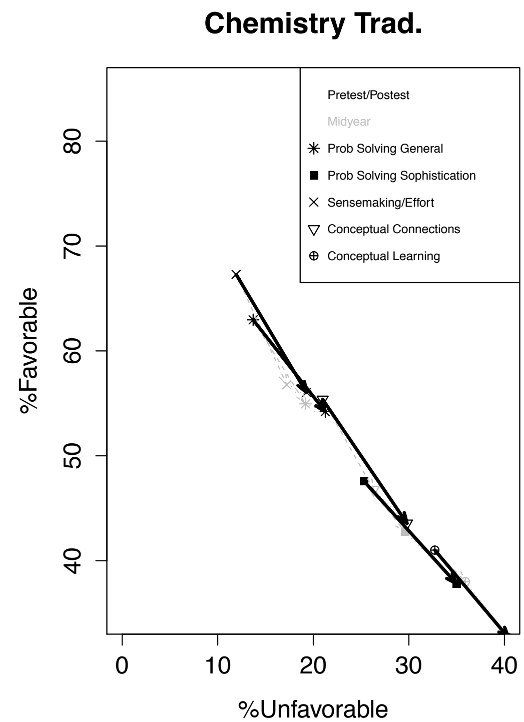
Table III: Math reasoning scores: percentage correct and p-values from significance test reported for the pretest, mid-year test and the end of year test.

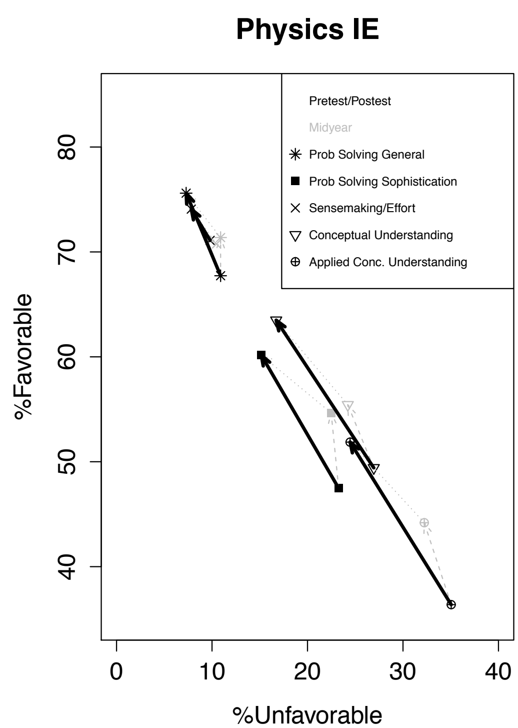
	Pre	*Mid*	*Post*	*Repeated t-test*
Mechanics Trad/IE ($n_{matched}$=447)	58	52	57	p-value $< 10^{-4}$ pre-mid; p-value $< 10^{-3}$ mid-post; p-value =0.79 pre-post
Chemistry Trad ($n_{matched}$=416)	59	52	57	p-value $< 10^{-6}$ pre-mid; p-value $< 10^{-3}$ mid-post; p-value = 0.45 pre-post
Mechanics IE (underprepared) ($n_{matched}$=90)	43	43	50	p-value = 0.91 pre-mid; p-value <0.005 pre-post

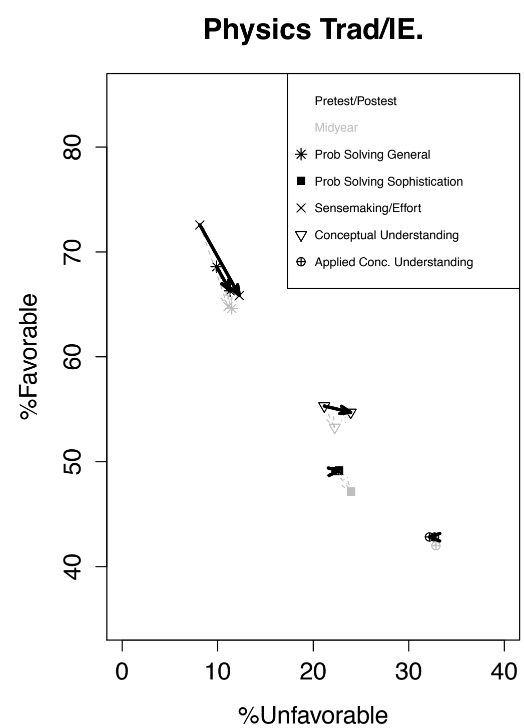
Attitudes and Beliefs Survey

Figure 1 shows the CLASS results for all three courses, labeled "Chem Trad" for the traditionally taught chemistry course, "Phys Trad/IE" for the mostly traditionally taught physics course and "Phys IE" for the transformed interactive engagement physics course for mathematically underprepared students. The data is clustered into the categories defined by the survey authors (Adams et al., 2006). For clarity we select only the 5 categories that are directly related to students' attitudes and beliefs about mathematization in physics: Problem Solving General, Problem Solving Sophistication, Sensemaking/Effort, Conceptual Understanding(Physics), Applied Conceptual Understanding (Physics), Conceptual Connections(Chemistry) and Conceptual Learning (Chemistry). As is common practice for representing these data, we plot the percentage of the students who agreed with the experts, called "favorable" on the vertical axis versus the percentage who disagree, called "unfavorable". Since being neutral is an option, those two scores do not, in general, add up to 100%. Pre-test and post-test are represented by black icons, and the mid year measurement is in gray. The gray, dashed arrows point from pre-test to mid-year, and mid-year to post-post test. The bold black arrows point from pretest to posttest. A movement to becoming more expert-like is represented by change arrows that point up and to the left on the graph. Arrows that point down and to the right represent degradation, or a change in attitudes and beliefs to becoming less expert-like.

Figure 1: CLASS result, by categories associated with mathematization. The favorable scores are plotted on the vertical axis and the unfavorable on the horizontal axis. Improvement is represented by movement up and to the left. $n_{PhysTrad}$=172, n_{PhysIE}=64, n_{Chem}=120







Discussion

There are several interesting patterns to note in the data regarding the measures of reasoning about linear relationships. The first is the glaring result that even though students may test very well on the Math SAT, their algebraic reasoning is likely to be much more primitive than their instructors think it is. The average pretest scores are less than 60% on questions that seem to be at the level of early secondary school algebra. Even more striking is that students' algebraic reasoning does not improve after having taken science courses that make heavy use of algebraic reasoning in the context of their discipline, regardless of the discipline. In addition, we report elsewhere that by altering surface features of the questions we see effects on the *robustness* of student reasoning. We see that the *robustness*, i.e. consistent use of reasoning across contexts and representations, of students' algebraic reasoning is context dependent, being less robust with abstract quantities like electric charge. We report also that the robustness is dependent on the complexity of the numbers used, being less robust with decimals than whole numbers, and much less robust with variables (Brahmia, Kanim, & Boudreaux, 2015).

Regarding the patterns of differences between the more traditionally taught courses and the course that focuses on developing mathematical reasoning, we see patterns with effects on a small scale. The students in the traditionally taught courses (that assume a conceptual understanding of algebra) actually reason less well at the end of the first semester. This deficit is rectified in the second semester bringing the students back to their starting point. Attending to the development of mathematization in the context of physics appears to do no harm to the students in the modified course during the first semester as their scores do not change, and helps them in the long run. None of these changes represent a very large *normalized gain* (percentage of what needs to be gained [post-pre]/[100%-pre]), which implies that improving students' algebraic reasoning in the context of physical science is a very challenging pedagogical problem.

Regarding attitudes and beliefs, students in the completely IE course clearly benefit from the targeted focus on the development of mathematical reasoning, and end up much more expert-like in there attitudes and beliefs about mathematization in physics than any of the students, and gained the most since they also started out much less expert-like than their engineering peers. The data from the traditional chemistry course score show trends similar to the published research regarding traditional courses. It is interesting that the students in both physics courses show less degradation of their attitudes than in their chemistry course. This may be because both physics courses involve at least some level of interactive engagement, while the chemistry course is an instructor-focused, lecture course only. We note similar trends on concept inventories when we look at pre and post test measurements, which is reported on in detail elsewhere (Brahmia, Boudreaux and Kanim, 2015). Comparing these three courses we see FCI normalized gains of 0.51 in the IE course, and 0.22 in the more traditional physics course, and essentially no gain on the CCI in the chemistry course.

Conclusion and Implications for Instruction

Mathematization of the physical sciences involves mathematical sensemaking, usually with abstract physical quantities. It involves attitudes about the use of mathematics, and fundamentally an appreciation for the utility of thinking mathematically. And it involves a facility and tendency to represent ideas and processes using mathematical constructs. We propose that student algebraic reasoning resources are fairly primitive, and that physics and chemistry are optimum disciplines for developing them further. We provide evidence in this paper that the mathematization goals implicit in the physical science courses are largely not being met.

It is known in mathematics education research that reasoning with quantity is challenging for students; it poses a significant challenge in college-level introductory physical science courses where students are bombarded with new and abstract quantities throughout their courses. We present evidence that even curricula shown to be highly effective at meeting many of the instructional goals of introductory physics are not as effective as they could be at activating reasoning resources that students have already developed, and these curricula often result in no or small improvement in students becoming more expert-like in the mathematization of physics. There has been a longstanding challenge in the physics education and chemistry education research communities to obtain anything other than negative or zero gains on the CLASS. By targeting students' mathematical sensemaking we present here a clear signal that students benefit and that they become more expert-like in their attitudes and beliefs about the role and utility of mathematical thinking in the physical sciences.

This work represents a preliminary step into understanding students' mathematization of the physical sciences. In order to move forward, we first must clearly understand what our instructional goals are for helping students develop mathematization – what does it mean to "think like a physical scientist"? There is much work to be done in better understanding the timeline of developing expert-like mathematical sensemaking. And even once we know what we hope students will learn, how can we better integrate this kind of reasoning development into the courses we teach?

Bibliography

Adams, W. K., Perkins, K. K., Podolefsky, N. S., Dubson, M., Finkelstein, N. D., & Wieman, C. E. (2006). New instrument for measuring student beliefs about physics and learning physics: The Colorado Learning Attitudes about Science Survey. *Physical Review Special Topics-Physics Education Research*, *2*(1), 010101.

Boudreaux, A., Kanim, S. E., & Brahmia, S. W. (2015). Student facility with ratio and proportion: Mapping the reasoning space in introductory physics (in preparation)

Brahmia, S. W., Boudreaux, A., & Kanim, S. E. (2015). Developing mathematization with Physics Invention Tasks (in preparation). http://inventiontasks.physics.rutgers.edu

Brahmia, S. W., Kanim, S. E., & Boudreaux, A. (2015). Obstacles to Mathematizaiton (in preparation)

Brahmia, S. (2008). *Improving Learning for Underrepresented Groups in Physics for Engineering Majors*. Proceedings from PER Conference Invited Paper, Edmonton, Canada.

Brewe, E., Traxler, A., de, l. G., Jorge, & Kramer, L. H. (2013). Extending positive CLASS results across multiple instructors and multiple classes of Modeling Instruction. *Phys. Rev. ST Phys. Educ. Res.*,*9, 2*, 020116.

College Board. (2015) Retrieved from https://bigfuture.collegeboard.org/find-colleges

Elby, A., Frederiksen, J., Schwarz, C., & White, B. (2003). The idea behind EBAPS. Retrieved from http://www2.physics.umd.edu/~elby/EBAPS/idea.htm

Halloun, I., & Hestenes, D. (1998). Interpreting VASS dimensions and profiles for physics students. *Science & Education*. Retrieved from http://umdberg.pbworks.com/w/file/fetch/38478710/IntrVASS.pdf

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The physics teacher*, *30*(3), 141-158.

Journal of Chemical Education. Chemical Concepts Inventory. (2015) Retrieved from http://www.jce.divched.org/jcedlib/qbank/collection/CQandChP/CQs/ConceptsInventory/CCIIntro.html

Lawson, A. E. (1978). The development and validation of a classroom test of formal reasoning. *Journal of Research in Science Teaching*, *15*(1), 11-24.

Lindsey, B. A., Hsu, L., Sadaghiani, H., Taylor, J. W., & Cummings, K. (2012). Positive attitudinal shifts with the Physics by Inquiry curriculum across multiple implementations. *Physical Review Special Topics-Physics Education Research*, *8*(1), 010102.

Otero, V. K., & Gray, K. E. (2008). Attitudinal gains across multiple universities using the Physics and Everyday Thinking curriculum. *Physical Review Special Topics-Physics Education Research*, *4*(2), 020104.

Rebello, N. S., Cui, L., Bennett, A. G., Zollman, D. A., & Ozimek, D. J. (2007). Transfer of learning in problem solving in the context of mathematics and physics. *Learning to solve complex scientific problems*, 223-246.

Redish, E. F., Saul, J. M., & Steinberg, R. N. (1998). Student expectations in introductory physics. *American Journal of Physics*, *66*(3), 212-224.

Redish, E., & Kuo, E. (2015). Language of Physics, Language of Math: Disciplinary Culture and Dynamic Epistemology. *Science & Education*, 1-30. doi:10.1007/s11191-015-9749-7

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. *Research companion to the principles and standards for school mathematics*, 95-113.