Two-Crested Stokes Waves

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Abstract

We study two-crested traveling Stokes waves on the surface of an ideal fluid with infinite depth. Following Chen & Saffman (1980), we refer to these waves as class II Stokes waves. The class II waves are found from bifurcations from the primary branch of Stokes waves away from the flat surface. These waves are strongly nonlinear, and are disconnected from small-amplitude solutions. Distinct class II bifurcations are found to occur in the first two oscillations of the velocity versus steepness diagram. The bifurcations in distinct oscillations are not connected via a continuous family of class II waves. We follow the first two families of class II waves, which we refer to as the secondary branch (that is primary class II branch), and the tertiary branch (that is secondary class II branch). Similar to Stokes waves, the class II waves follow through a sequence of oscillations in velocity as their steepness rises, and indicate the existence of limiting class II Stokes waves characterized by a 120 degree angle at every other wave crest.

Introduction

Nonlinear traveling waves on the surface of a 2D ideal fluid are called Stokes waves, first described by Stokes [1847, 1880]. Such waves retain their shape in the reference frame moving with their velocity. Stokes waves, and in particular the ones with one crest per period have been studied extensively. Following Chen and Saffman [1980] we refer to such waves as class I waves. Specifically, the existence of such waves was shown by Nekrasov [1921] and Levi-Civita [1925]. It was proven by Toland [1978] that there exists a limiting class I Stokes wave, and for the proof that an angle of 120 degrees forms at its crest, see Amick, Fraenkel, and Toland [1982], Plotnikov [1982]. The study of the stability of Stokes waves has a long history, see for example Benjamin and Feir [1967], Longuet-Higgins and Tanaka [1997], Nguyen and Strauss [2023], Berti, Maspero, and Ventura [2022], Deconinck, Dyachenko, and Semenova [2024]. In the work Deconinck, Dyachenko, Lushnikov, and Semenova [2023], it was shown

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that large, near limiting waves are unstable with respect to disturbances localized at their crests. In particular, it was found that the dominant disturbances are either co-periodic or have twice the period of the wave. The subject of this paper is traveling waves that have two crests per wavelength referred to as the class II Stokes waves.

We consider a 2D ideal fluid with a free surface and infinite depth. The flow is assumed to be potential with fluid velocity $\mathbf{v} = \nabla\phi$, where ϕ is the velocity potential. The only force acting on the fluid is gravity, and periodic traveling surface waves are considered. The free surface is described by a 1D curve $y = \eta(x,t)$, where x and y are horizontal and vertical spatial variables, and t is time. The motion of the free surface is described by the Laplace equation in the fluid domain $\mathcal{D} = \{(x,y) | -\pi < x < \pi, -\infty < y < \eta(x,t)\}$ and nonlinear time-dependent boundary conditions,

$$\begin{cases} \Delta\varphi = 0 \text{ in } \mathcal{D}, \\ \eta_t = -\varphi_x\eta_x + \varphi_y \text{ at } y = \eta(x,t), \\ \varphi_t + \frac{1}{2}(\nabla\varphi)^2 + g\eta = 0 \text{ at } y = \eta(x,t), \end{cases} \qquad \varphi_y\big|_{y\to-\infty} = 0.$$

Here g is the acceleration due to gravity. We use the conformal transformation approach

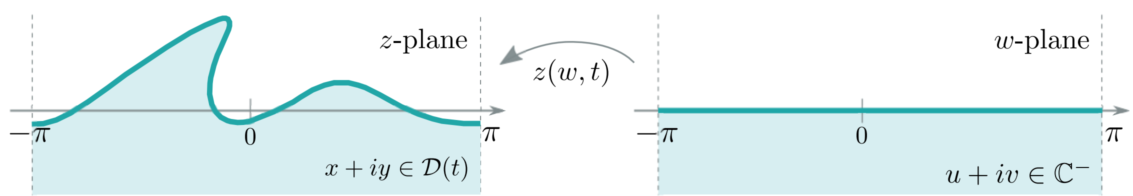

Figure 1: The region in the w plane $(u,v) \in [-\pi,\pi] \times (-\infty, 0]$ is mapped into the domain occupied by the fluid in the $(x,y)-$plane $(x,y) \in [\pi,\pi] \times (-\infty, \eta(x,t)]$. The line $v = 0$ (blue line) is mapped onto the free surface $\eta(x,t)$.

described in Ovsyannikov [1973] and later in Tanveer [1991], Zakharov, Kuznetsov, and Dyachenko [1996] (see also Dyachenko, Lushnikov, and Korotkevich [2016] for the periodic problem). In the conformal variables the equations of motion can be found from the variational principle, and have the form,

$$y_t x_u - y_u x_t = -\hat{H}\psi_u, \tag{1}$$

$$x_t\psi_u - x_u\psi_t - \hat{H}\left[y_t\psi_u - y_u\psi_t\right] = g\left(x_u y - \frac{1}{2}\hat{H}\partial_u y^2\right), \tag{2}$$

where $\hat{H}$ is the Hilbert transform described below, and ψ is the value of the potential φ on the surface. The conformal map $z(w,t) = x(w,t) + iy(w,t)$ is a time-dependent holomorphic function of $w = u + iv$ that maps the lower complex half-plane $(u,v) \in (-\pi,\pi) \times (-\infty, 0]$ into the fluid domain $\mathcal{D}$ as presented in Figure 1. The equations (1)–(2) are posed on the real line $v = 0$, and $\hat{H}$ is the periodic Hilbert transform that is defined by,

$$\hat{H}f(u) = \frac{1}{2\pi} \fint_{-\pi}^{\pi} f(u') \cot\frac{u'-u}{2} du'. \tag{3}$$

In Fourier space, the Hilbert transform has the Fourier symbol $i\,\mathrm{sign}k$, or equivalently $\hat{H}e^{iku} = i\,\mathrm{sign}(k)\,e^{iku}$. The Hamiltonian of the fluid is expressed in the conformal variables as,

$$\mathcal{H} = \frac{1}{2}\int_{-\pi}^{\pi} \psi\hat{k}\psi\,du + \frac{g}{2}\int_{-\pi}^{\pi} y^2 x_u\,du, \tag{4}$$

where $\hat{k}$ is the Dirichlet-to-Neumann operator with $\hat{k} = -\hat{H}\partial_u$ for the lower complex half-plane.

We substitute a traveling wave ansatz $y = y(u-ct)$, $\psi = \psi(u-ct)$ with propagation velocity c into the equations (1)–(2), which yields the Babenko [1987] equation,

$$\left(c^2\hat{k} - g\right) y\,(u) - \frac{g}{2}\left[\hat{k}y^2\,(u) + 2y\,(u)\,\hat{k}y\,(u)\right] = 0. \tag{5}$$

Here the conformal map z and the potential ψ are uniquely determined from $y(u)$.

Waves with two distinct crests per wavelength have been computed by Chen and Saffman [1980] and are called class II waves (here II stands for the number of crests per period). In that paper, the authors also studied three-crested waves which are referred to as class III solutions. The bifurcation points from the family of one-crested Stokes waves to symmetric multi-period waves were also studied by Saffman [1980], Vanden-Broeck [1983], Longuet-Higgins [1985], Vanden-Broeck [2017]. In this text, we adopt the Chen and Saffman [1980] terminology and refer to the solutions of equation (5) with two crests of different amplitudes per period as class II waves. We also introduce notation for a Stokes wave with one crest per period as the solution from the main branch, or a class I wave. Non-symmetric waves and corresponding bifurcation points on the main branch of Stokes waves are discussed by Zufiria [1987], and recently by Wilkening and Zhao [2021].

Similarly to Dyachenko, Lushnikov, and Korotkevich [2014], we solve the equation (5) via the Newton-MINRES method (see Yang [2009]) in Fourier space. We have chosen to use the minimal residual method (MINRES) instead of the Conjugate-Gradient method because although the linearized Babenko operator is self-adjoint, it is not positive definite. We use the double period bifurcation points computed in Dyachenko and Semenova [2023] for 2π-periodic Stokes waves to find and compute traveling waves that have twice the period of the original one-crested Stokes waves. Since we consider class II waves with wavelength 2π, we rescale a "one crested Stokes wave" with wavelength 2π to have two identical crests of half amplitude in each interval of length 2π. Given a 2π-periodic one-crested Stokes wave with velocity c, and a vertical displacement $y(u)$, the wave of the form $\left(c/\sqrt{2}, y(2u)/2\right)$ is also a solution of the Babenko equation. It has exactly two identical crests on the interval $u \in [-\pi, \pi]$. When a Stokes wave is located at a double period bifurcation point, we can compute and study properties of 2π-periodic class II Stokes waves by means of the continuation method in the velocity parameter, c, exactly as we would continue from the flat water to the main branch of class I Stokes waves.

To compute class II waves, we use the continuation method for c and start from the velocity c^s. The initial guess for the Newton-MINRES iterations is chosen to be

$$y_0(u) = y^s(u) + \varepsilon f(u), \tag{6}$$

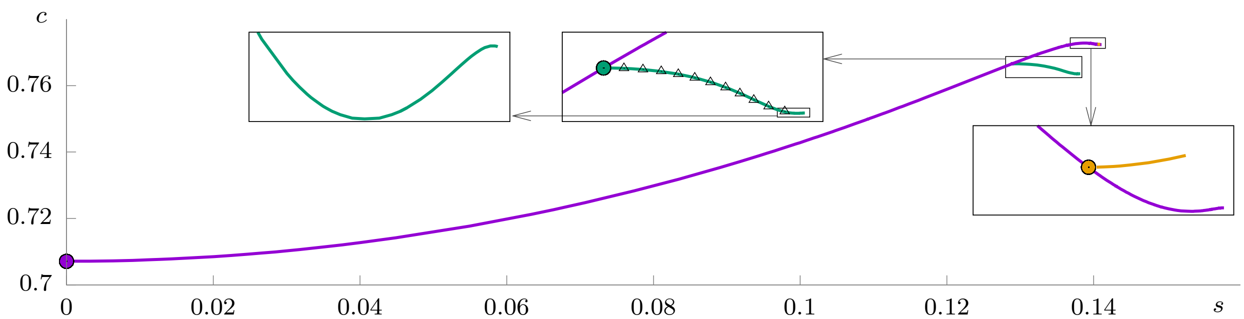

Figure 2: Oscillations in velocity c as a function of steepness s. The purple curve corresponds to Stokes waves from the primary branch (one crested Stokes waves per wavelength); The green curve represents the class II Stokes waves with period 2π bifurcating from s_1 (secondary branch); The gold curve is the second family of the class II waves (tertiary branch) bifurcation from s_2. Black triangles are the data for class II waves from Chen and Saffman [1980] which fits well onto the secondary branch (green curve).

where $(c^s, y^s(u))$ is the solution of the Babenko equation (5) from the primary branch at the period-doubling bifurcation point. Here $f(u)$ is the eigenfunction corresponding to the zero eigenvalue of the linearized Babenko operator (see Dyachenko and Semenova [2023] for details), and ε is a small number (we set ε to be 10^{-2}). With this choice of the initial guess, the Newton-MINRES method converges to a class II traveling wave of the Babenko equation (5). The sign of ε dictates which crest (left or right) becomes the taller one past the bifurcation point, however the resulting solution branches are equivalent class II waves and are identical after a horizontal shift by one half of a wavelength. Hence, without loss of generality the primary branch solution is perturbed via (6) with $\varepsilon > 0$. The auxiliary conformal transformation from Lushnikov, Dyachenko, and Silantyev [2017] is used for computations of steep waves to speed up convergence of the Fourier series. We observe that a corner of 120 degrees tends to form at the taller crest as we follow along the branch, see Fig. 4.

Waves Profiles and Discussion

Chen and Saffman [1980] computed the first double period-bifurcation point $s_1^{CS} = 0.1289$ on the primary branch of the 2π-periodic class I Stokes waves. The second bifurcation point $s_2^{VB} = 0.14$ was computed by Vanden-Broeck [1983]. Recently, Vanden-Broeck [2017] computed the third bifurcation point together with some class II waves from the quaternary branch. In this work, we use double-period bifurcation points $s_1 = 0.128903$, $s_2 = 0.140487$, and $s_3 = 0.141032049$ computed in Dyachenko and Semenova [2023] and the conformal mapping approach described above to study the class II waves from the secondary and tertiary branches. We define the steepness parameter to be $s = H/\pi$ which is the ratio of the distance between the taller crest and the deeper trough H divided by π. This parameter differs by a multiple of $1/2$ from the conventional definition of steepness which is the ratio of

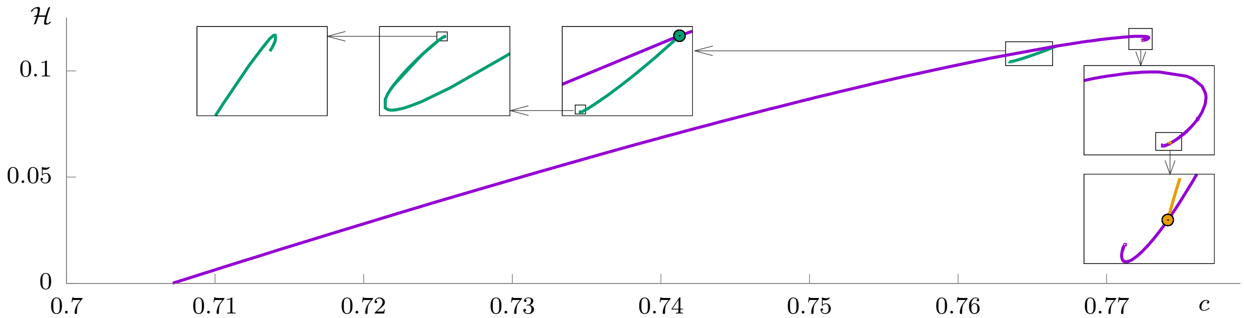

Figure 3: The Hamiltonian $\mathcal{H}$ as a function of velocity c. Green and gold circles mark the double-period bifurcation points from the primary branch of class I Stokes waves (purple curve). Class II waves from the secondary and tertiary branch are shown by green and gold curves respectively. The insets show winding of the primary (purple) and secondary (green) branches. These spirals indicate extrema in both the Hamiltonian and the velocity as functions of steepness for these 2 curves.

the height to the wavelength (in this case it is 2π). The reason for this choice is the ability to compare the values of s to the steepness of extensively studied one-crested 2π-periodic Stokes waves.

In Figure 2, we show the velocity c as a function of steepness s. The purple curve represents the class I Stokes waves from the primary branch with oscillations in c as the limiting wave is approached, as we know from the works of Longuet-Higgins and Fox [1978], Dyachenko et al. [2014]. The green curve corresponds to the branch of 2π-periodic waves of class II where one of the crests is taller than the other. We refer to this family of class II Stokes waves as the secondary branch. These waves bifurcate from the double-period bifurcation point $s_1 = 0.128903$ (marked by the green circle) computed in Dyachenko and Semenova [2023]. The black triangles are the data from Chen and Saffman [1980] which fits well with our results for the secondary branch except for the last three waves. We are able to extend the results of Chen and Saffman [1980] and Vanden-Broeck [2017], and compute even steeper waves from the secondary branch as can be seen in the two insets for the green curve that show oscillations in velocity as the steepness of waves increases. We compute 2 local extrema (first minimum and second maximum) in the green curve, and conjecture that there are infinitely many such extrema, qualitatively similar to the class I waves on the main branch. Also, we note that a corner of 120 degrees forms at the taller crest and a class II limiting wave is evident. The limiting value of velocity c for class II waves is $c_{lim}^{2nd} = 0.7635\ldots$ (with 4 digits of accuracy) which is different from the value $c_{lim}^{p} = 0.77236216478\ldots$ for the limiting Stokes wave from the primary branch, see Lushnikov et al. [2017]. The golden curve in Figure 2 represents the second family of two-crested waves that form the tertiary branch and bifurcates from the primary branch of Stokes waves at $s_2 = 0.140487$. We conjecture that this branch also has infinitely many oscillations in the velocity as a function of the wave steepness and has its own limiting wave. The value of the velocity for this limiting wave will be different from the values for primary (purple curve) and secondary (green curve)

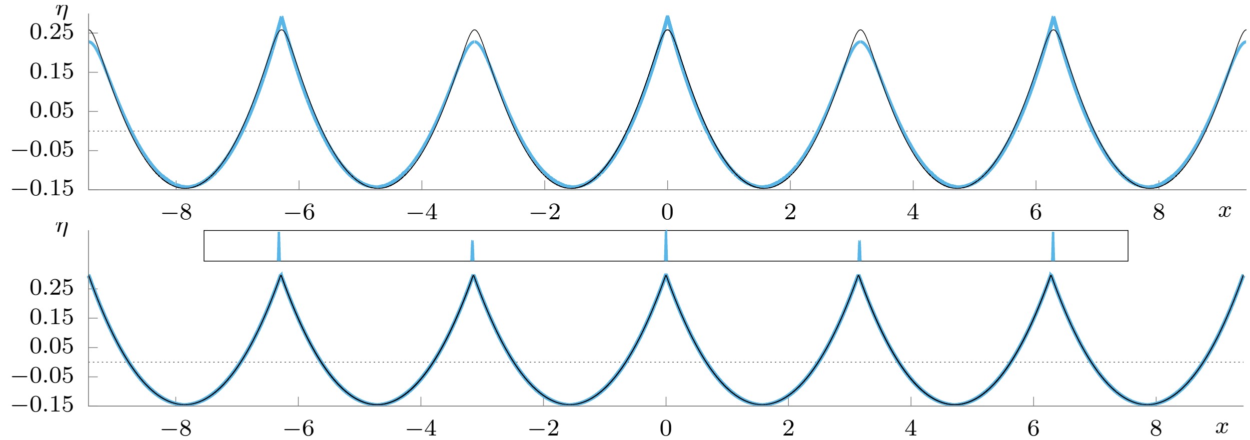

Figure 4: Both panels show 3 periods of waves. The black lines are the Stokes waves from the primary branch, s_1 (top panel) and s_2 (bottom panel). (Top Panel) Profile of the class II wave $(s, c) = (0.1381671, 0.7635778)$ (blue curve) from the secondary branch (originating from s_1). (Bottom Panel) Profile of the class II wave $(s, c) = (0.1408973, 0.77245)$ (blue curve) from the tertiary branch (originating from s_2). The difference in the crests amplitudes is shown in the inset.

branches. Moreover, we know that there is at least one more double-period bifurcation point $s_3 = 0.141032049$ computed in Dyachenko and Semenova [2023], and there is a third family of class II waves (the quaternary branch) originating from it, see Vanden-Broeck [2017]. Furthermore, we conjecture that there are infinitely many such families/branches of class II waves (since it is conjectured in Dyachenko and Semenova [2023] that there are infinitely many zero eigenvalues of the linearized Babenko operator). They all form a corner at one or more crests as their respective limiting waves are approached.

The plot of the Hamiltonian $\mathcal{H}$ as a function of velocity c is a spiral as we present in Figure 3. The purple line represents Stokes waves from the primary branch, and it spirals around the center $(c^p_{lim}, \mathcal{H}^p_{lim})$ corresponding to the limiting class I Stokes wave. When shown as the function of wave steepness s the Hamiltonian oscillates as the limiting wave is approached, see Longuet-Higgins and Fox [1977], Korotkevich, Lushnikov, Semenova, and Dyachenko [2023], and the extrema of the velocity and the Hamiltonian correspond to the points where the tangent line to the spiral in Figure 3 is vertical and horizontal respectively. The extrema of the Hamiltonian occur before the extrema of the velocity and correspond to a clockwise spiral. The green and gold circles on the primary branch mark the double-period bifurcation points to the secondary (green) and tertiary (gold) branches of class II waves respectively. The first two extrema in the Hamiltonian of the secondary branch are captured by the winding of the green curve and presented in the three insets. We conjecture that more bifurcation points exist on the secondary branch as the curve spirals towards the class II limiting wave $\left(c^{2nd}_{lim}, \mathcal{H}^{2nd}_{lim}\right)$. Similar to the diagram for class I waves, we note that the local extrema in the Hamiltonian occur at a smaller steepness than the extrema in the velocity. We expect the tertiary branch to have similar winding behavior to the primary and

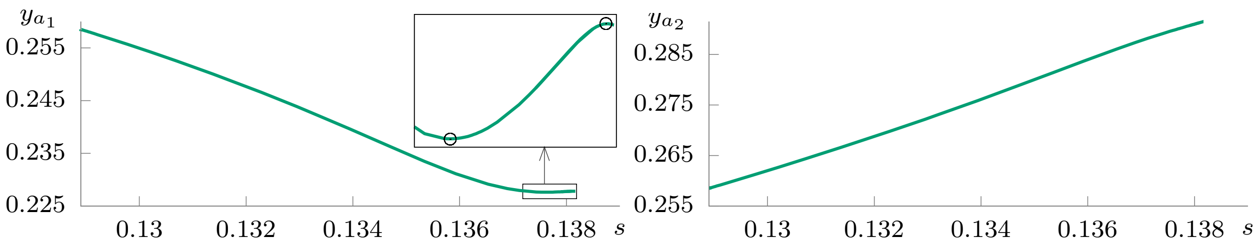

Figure 5: In both panels we plot the crests amplitudes of class II waves from the secondary branch as functions of the waves steepness. The amplitude of the smaller crest (Left Panel) oscillates (increases and decreases) and the amplitude of the crest that forms an angle of 120 degrees (Right Panel) monotonically increases as the limiting class II wave is approached. The first 2 extremizer of y_{a_1} are $s^* = 0.137619$ and $s^* = 0.13814$ (marked by black circles).

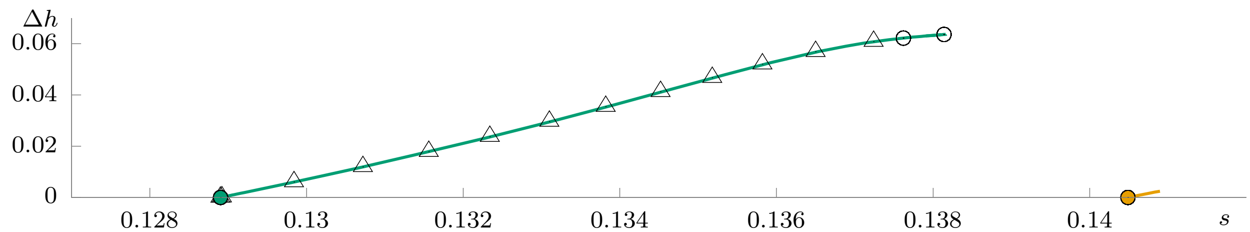

Figure 6: The height difference Δh between two crests of class II waves versus steepness s for the secondary branch (green curve) and tertiary branch (golden curve). Black triangles correspond to data from Chen and Saffman [1980]. The green and golden circles mark the first and second double period bifurcation points (s_1 and s_2) respectively.

secondary branches, and spiral to its own limiting wave.

In Figure 4, we present profiles of class II waves (blue curves) from the secondary (top panel) and tertiary (bottom panel) branches. The main branch solutions with steepness s_1 and s_2 correspond to Stokes waves that bifurcate to the class II waves, and are marked with black lines in the top and bottom panels of Figure 4 respectively. Three periods of class II waves with $s = 0.138167\ldots$ (top panel) and $s = 0.140897\ldots$ (bottom panel) are shown. The inset in the bottom panel is a zoom-in to the crest region of that wave, and is necessary to display the amplitude difference which is otherwise visually indistinguishable. We track the amplitudes of the crests of class II waves from the secondary branch in Figure 5. One of the crests grows monotonically and forms a corner (right panel) while the amplitude of the other one oscillates (left panel). We conjecture that there are infinitely many such oscillations in the amplitude of the other crest, and it does not form a corner as the steeper crest approaches the limiting shape. The height difference between the two crests Δh (green curve in Figure 6) grows as the steepness of the waves increases. The black triangles correspond to the data from Chen and Saffman [1980], and they are marked to offer a direct comparison with preceding studies of double-period bifurcations (green and gold circles are the double period bifurcations).

Conclusion

We have computed and studied class II waves (traveling waves with two crest of different amplitudes per period) thus expanding observations of Chen and Saffman [1980], Vanden-Broeck [2017]. Two branches of class II waves have been computed. The secondary branch bifurcates from $s_1 = 0.128903$, and it was shown that the velocity c and the Hamiltonian $\mathcal{H}$ (for the secondary branch) have oscillations as the limiting class II wave $c_{lim}^{2nd} = 0.7635$ is approached. The tertiary branch of two crested wave bifurcating from $s_2 = 0.140487$ is shown. We conjecture that the velocity c and the Hamiltonian $\mathcal{H}$ oscillates for steeper waves from this branch just as it does for the class I waves, and the details of this behaviour is left for future work. Such recurrent behaviour, together with the conjecture in Dyachenko and Semenova [2023] about an infinite number of zero eigenvalues of the linearized Babenko operator, may be viewed as a reinforcement that an infinite number of bifurcating points to class II waves is located at the primary branch of Stokes waves, see Vanden-Broeck [2017]. The stability of the class II waves and comparison with results from Deconinck et al. [2023] for class I Stokes waves from the primary branch remains open and is left for future work.

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References

C.J. Amick, L.E. Fraenkel, and J.F. Toland. On the Stokes conjecture for the wave of extreme form. *Acta Mathematica*, 148(1):193–214, 1982.

K.I. Babenko. Some remarks on the theory of surface waves of finite amplitude. In *Doklady Akademii Nauk*, volume 294, pages 1033–1037. Russian Academy of Sciences, 1987.

T.B. Benjamin and J.E. Feir. The disintegration of wave trains on deep water. *Journal of Fluid Mechanics*, 27(3):417–430, 1967.

M. Berti, A. Maspero, and P. Ventura. Full description of Benjamin-Feir instability of Stokes waves in deep water. *Inventiones mathematicae*, 230(2):651–711, 2022.

B. Chen and P.G. Saffman. Numerical evidence for the existence of new types of gravity waves of permanent form on deep water. *Studies in Applied Mathematics*, 62(1):1–21, 1980.

B. Deconinck, S.A. Dyachenko, P.M. Lushnikov, and A. Semenova. The dominant instability of near-extreme Stokes waves. *Proceedings of the National Academy of Sciences*, 120(32): e2308935120, 2023.

B. Deconinck, S.A. Dyachenko, and A. Semenova. Self-similarity and recurrence in stability spectra of near-extreme stokes waves. *Journal of Fluid Mechanics*, 995:A2, 2024. doi: 10.1017/jfm.2024.626.

S.A. Dyachenko and A. Semenova. Quasiperiodic perturbations of Stokes waves: Secondary bifurcations and stability. *Journal of Computational Physics*, page 112411, 2023.

S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich. The complex singularity of a Stokes wave. *JETP Letters*, 98(11):675–679, 2014.

S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich. Branch cuts of Stokes wave on deep water. Part I: numerical solution and Padé approximation. *Studies in Applied Mathematics*, 137(4):419–472, 2016.

A.O. Korotkevich, P.M. Lushnikov, A. Semenova, and S.A. Dyachenko. Superharmonic instability of Stokes waves. *Studies in Applied Mathematics*, 150(1):119–134, 2023.

T. Levi-Civita. Détermination rigoureuse des ondes permanentes d'ampleur finie. *Mathematische Annalen*, 93(1):264–314, 1925.

M.S. Longuet-Higgins. Bifurcation in gravity waves. *Journal of Fluid Mechanics*, 151:457–475, 1985.

M.S. Longuet-Higgins and M.J.H. Fox. Theory of the almost-highest wave: the inner solution. *Journal of Fluid Mechanics*, 80(4):721–741, 1977.

M.S. Longuet-Higgins and M.J.H. Fox. Theory of the almost–highest wave. Part 2. Matching and analytic extension. *Journal of Fluid Mechanics*, 85:769–786, 1978.

M.S. Longuet-Higgins and M. Tanaka. On the crest instabilities of steep surface waves. *Journal of Fluid Mechanics*, 336:51–68, 1997.

P.M. Lushnikov, S.A. Dyachenko, and D.A. Silantyev. New conformal mapping for adaptive resolving of the complex singularities of Stokes wave. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 473(2202):20170198, 2017.

A.I. Nekrasov. On waves of permanent type I. *Izv. Ivanovo-Voznesensk. Polite. Inst.*, 3: 52–65, 1921.

H.Q. Nguyen and W.A. Strauss. Proof of Modulational Instability of Stokes Waves in Deep Water. *Communications on Pure and Applied Mathematics*, 76(5):1035–1084, 2023.

L.V. Ovsyannikov. Dynamika sploshnoi sredy, Lavrentiev Institute of Hydrodynamics. *Sib. Branch Acad. Sci. USSR*, 15:104, 1973.

P.I. Plotnikov. Justification of the Stokes conjecture in the theory of surface waves (in Russian). *Dinamika Sploshnoi Sredy*, 57:4176, 1982.

PG Saffman. Long wavelength bifurcation of gravity waves on deep water. *Journal of Fluid Mechanics*, 101(3):567–581, 1980.

G.G. Stokes. On the theory of oscillatory waves. *Transactions of the Cambridge Philosophical Society*, 8:441, 1847.

G.G. Stokes. On the theory of oscillatory waves. *Mathematical and Physical Papers*, 1:197, 1880.

S. Tanveer. Singularities in water waves and Rayleigh–Taylor instability. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 435(1893): 137–158, 1991.

J.F. Toland. On the existence of a wave of greatest height and Stokes's conjecture. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 363(1715): 469–485, 1978.

J.-M. Vanden-Broeck. Some new gravity waves in water of finite depth. *Physics of Fluids*, 26(9):2385–2387, 1983.

J.-M. Vanden-Broeck. New families of pure gravity waves in water of infinite depth. *Wave Motion*, 72:133–141, 2017.

Jon Wilkening and Xinyu Zhao. Spatially quasi-periodic water waves of infinite depth. *Journal of Nonlinear Science*, 31(3):52, 2021.

J. Yang. Newton-conjugate-gradient methods for solitary wave computations. *Journal of Computational Physics*, 228(18):7007–7024, 2009.

V.E. Zakharov, E.A. Kuznetsov, and A.I. Dyachenko. Dynamics of free surface of an ideal fluid without gravity and surface tension. *Fizika Plasmy*, 22:916–928, 1996.

J.A. Zufiria. Non-symmetric gravity waves on water of infinite depth. *Journal of Fluid Mechanics*, 181:17–39, 1987.