C Proof of part (ii), Theorem 1 for the case of arbitrary finite signal sets (and hence bounded likelihood ratios)

PRELIMINARIES

Let the seller and buyer signals be drawn from arbitrary finite sets S_1 and S_2 respectively. Distinct signals are associated with distinct likelihood ratios: for $i = 1, 2$, if s and $\tilde{s}$ are distinct elements of S_i then $L_i(s) \neq L_i(\tilde{s})$.

For $i = 1, 2$, define s_i^a and s_i^b respectively as the most pro-a and pro-b signals in S_i, i.e.,

$$s_i^a = \arg\max_{s \in S} L_i(s)$$
$$s_i^b = \arg\min_{s \in S} L_i(s).$$

We assume the buyer has at least some information (trade is clearly impossible otherwise), in the sense that S_2 contains at least two elements. Hence $L_2(s_2^a) > 1 > L_2(s_2^b)$. We assume that the quality of the seller's (agent 1's) information is weakly better than the buyer's (agent 2) in the following mild sense:

$$\frac{\Pr\left(s_1^a s_2^b | a\right)}{\Pr\left(s_1^a s_2^b | b\right)} \geq \frac{\Pr\left(s_1^b s_2^a | a\right)}{\Pr\left(s_1^b s_2^a | b\right)}. \tag{33}$$

That is, if agents receive conflicting and extreme signals, the seller's signal is weakly more indicative of the true state. Equal information quality is, of course, a special case. Focusing on this case makes trade harder to obtain compared to the opposite case. Moreover, for most applications it is natural to assume that the existing owner knows more about the asset than does a potential buyer.[35]

[35]Note that it is quite possible for the expected value of the asset to the buyer to exceed the expected value of the asset to the seller, even if both observe only their own signals s_2 and s_1

In all other respects the economy is unchanged from the main text.

Because the state space $\Omega = \{a, b\} \times S_1 \times S_2$ is now finite, it is easier to work with partitional representations of agents' information. For $i = 1, 2$, we write $\mathcal{P}_i$ and $\mathcal{P}_i^{\kappa,\pi}$ for the partitions corresponding to pre-trade information $\mathcal{F}_i$ and post-trade information $\mathcal{F}_i^{\kappa,\pi}$.

Proof of part (ii), Theorem 1

In the finite signal case, the following circumstance is non-generic: the buyer learns nothing, yet places exactly the same value on the asset as the seller (regardless of what the seller learns about the buyer's signal). We restrict attention to the generic case in which this does arise.

The proof is by contradiction, and is similar to the proof in the main text with unbounded likelihood ratios. The main difference is that the proof of the analogue of inequality (24) is more involved — see Lemma 3 below.

respectively. Consider the following. The unconditional probability of $\theta = a$ is 1/2; the action set is $\mathcal{X} = \{A, B\}$; the asset payoffs are $v(A, a) = 2$, $v(B, b) = 1$, $v(A, b) = v(B, a) = 0$; and the signal sets for both agents are binary: $S_i = \left\{s^a, s^b\right\}$. For the seller, $\Pr(s_1 = s^a|a) = 0.97$ and $\Pr\left(s_1 = s^b|b\right) = 0.73$. For the buyer, $\Pr(s_2 = s^a|a) = \Pr\left(s_2 = s^b|b\right) = 0.9$. It is easily verified that agent i takes action A if $s_i = s^a$ and action B if $s_i = s^b$, for $i = 1, 2$. Note that

$$\frac{\Pr\left(s_1 = s^a, s_2 = s^b|a\right)}{\Pr\left(s_1 = s^a, s_2 = s^b|b\right)} = \frac{.97}{.27}\frac{.1}{.9} > \frac{.03}{.73}\frac{.9}{.1} = \frac{\Pr\left(s_1 = s^b, s_2 = s^a|a\right)}{\Pr\left(s_1 = s^b, s_2 = s^a|b\right)},$$

so that the seller's signal is more informative than the buyer's, in the sense that condition (33) is satisfied. Observe that while the seller's signal $s_1 = s^b$ is more pro-b than the buyer's signal $s_2 = s^b$, the seller's signal $s_1 = s^a$ is *less* pro-a than the buyer's signal $s_2 = s^a$. Concretely, the seller's signal $s_1 = s^a$ is not very informative because it is observed often when $\theta = b$. As such, the seller often makes the wrong decision in $\theta = b$. In contrast, the buyer is less likely to observe $s_2 = s^a$ in $\theta = b$, and so makes the wrong decision in $\theta = b$ less often. Conversely, he makes the wrong decision in $\theta = a$ more often than the seller does. However, the absolute cost of the seller's mistakes exceeds that of the buyer's (even though mistakes are more costly in $\theta = a$ than $\theta = b$).

Suppose to the contrary that there is a trade (κ, π) such that common knowledge of strictly positive gains from trade exists, and in which the buyer learns nothing whenever he acquires the asset. Let (θ, s_1, s_2) be a state in which the buyer acquires the asset, and $p = \pi(\theta, s_1, s_2)$ be the price paid. Since the buyer learns nothing, trade must occur at the same terms over $\{a, b\} \times S_1 \times \{s_2\}$. It follows that the subset of the state space in which trade occurs at price p is of the form $\{a, b\} \times S_1 \times S_2^T$, where S_2^T is a subset of S_2.[36]

For all $s_2 \in S_2^T$ the buyer's valuation exceeds the price he pays, p. Since the buyer does not learn anything,

$$p \leq V(\Pr(a|s_2)). \tag{34}$$

The seller's information partition after trade is $\mathcal{P}_1^{\kappa,\tau}$. Note that $\{a, b\} \times S_1 \times S_2^T$ is $\mathcal{P}_1^{\kappa,\tau}$-measurable since the seller learns at least the information conveyed by the trade. So the corresponding condition for the seller is that

$$p \geq V(Q). \tag{35}$$

for all elements $Q \in \mathcal{P}_1^{\kappa,\tau}$ such that $Q \subset \{a, b\} \times S_1 \times S_2^T$.

Suppose for now that we can find $Q, Q' \in \mathcal{P}_1^{\kappa,\tau}$ such that $Q, Q' \subset \{a, b\} \times S_1 \times S_2^T$ and $s_2 \in S_2^T$ such that

$$\Pr(a|Q) \geq \Pr(a|s_2) \geq \Pr(a|Q'), \tag{36}$$

i.e., a signal realization s_2 for the buyer such that sometimes the seller's information is more pro-a, and sometimes it is more pro-b — and all three pieces of information are associated with trade.

By the convexity of V, condition (36) implies that

$$\max\{V(\Pr(a|Q)), \Pr(a|Q')\} \geq V(\Pr(a|s_2)).$$

[36]Of course, trade may occur at a different price in some other subset of the state space.

Recall that we restrict attention to the generic case in which this inequality holds strictly. Consequently, the existence of s_2, Q, Q' is inconsistent with the trade conditions (34) and (35), and provides the required contradiction. Less formally, the asset is less valuable to an agent who is unsure about the true state than to one who is relatively confident about the true state. As such, the buyer's valuation at signal s_2 must be less than the seller's valuation at one of Q and Q'. Formally, this follows from the the convexity of the function V.

The proof is thus complete if we can show that there exist s_2, Q, Q' with these properties.[37] We establish:

Lemma 3. *Let the seller's and buyer's information partitions be $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively, and suppose there exists a subset S_2^T of the buyer's signal set S_2 such that (i) the buyer learns only his own signal when $s_2 \in S_2^T$, that is, $\{a, b\} \times S_1 \times \{s_2\} \in \mathcal{P}_2$ for all $s_2 \in S_2^T$; and (ii) $\{a, b\} \times S_1 \times S_2^T$ is measurable with respect to the seller's information $\mathcal{P}_1$. Then there exist $Q, Q' \in \mathcal{P}_1$ and $s_2 \in S_2^T$ such that $Q, Q' \subset \{a, b\} \times S_1 \times S_2^T$ and condition (36) holds.*

The intuition for Lemma 3 is as follows. By assumption, the informativeness of the seller's signal exceeds that of the buyer's, in the sense of condition (33). It follows that either (i) the seller's most pro-a signal is more pro-a than the buyer's most pro-a signal; or (ii) the seller's most pro-b signal is more pro-b than the buyer's most pro-b signal. In case (i), pick s_2 to be the most pro-a signal in S_2^T. Given this choice, it is straightforward to find seller information Q' that is more pro-b than s_2. Moreover, the fact that we are in case (i) implies that there is seller information Q that is more pro-a than s_2.

[37]Given our assumption that the seller's signal is weakly better than the buyer's, some readers may conjecture that a more direct proof is available: if the seller knows weakly more, the asset is on average more valuable to him than to the buyer, and so no trade is possible. However, this conclusion is not valid. The example of footnote 35 provides a counterexample.

Proof of Lemma 3: Recall that s_i^a and s_i^b are, respectively, the most pro-a and most pro-b of agent i's signals. We start by establishing the following minor result:

Lemma 4. *Let signals $s_2, s_2' \in S_2$ be a pair of buyer signals (possibly the same), and $\hat{S}_1$ and $\hat{S}_2$ signal subsets of S_1 and S_2 respectively. Then either $\{s_1^a\} \times \hat{S}_2$ is more pro-a than $\hat{S}_1 \times \{s_2'\}$ or $\{s_1^b\} \times \hat{S}_2$ is more pro-b than $\hat{S}_1 \times \{s_2\}$.*

Proof of Lemma 4: From the definitions of s_2^a, s_2^b and condition (33),

$$\frac{\Pr(s_1^a s_2|a)}{\Pr(s_1^a s_2|b)} \geq \frac{\Pr\left(s_1^a s_2^b|a\right)}{\Pr\left(s_1^a s_2^b|b\right)} \geq \frac{\Pr\left(s_1^b s_2^a|a\right)}{\Pr\left(s_1^b s_2^a|b\right)} \geq \frac{\Pr\left(s_1^b s_2'|a\right)}{\Pr\left(s_1^b s_2'|b\right)}.$$

Multiplying the first and last terms by $\frac{\Pr\left(\hat{S}_2|a\right)}{\Pr\left(\hat{S}_2|b\right)}\frac{\Pr\left(\hat{S}_1|a\right)}{\Pr\left(\hat{S}_1|b\right)}$ gives

$$\frac{\Pr(s_1^a|a)}{\Pr(s_1^a|b)}\frac{\Pr\left(\hat{S}_2|a\right)}{\Pr\left(\hat{S}_2|b\right)}\frac{\Pr(s_2|a)}{\Pr(s_2|b)}\frac{\Pr\left(\hat{S}_1|a\right)}{\Pr\left(\hat{S}_1|b\right)} \geq \frac{\Pr\left(s_1^b|a\right)}{\Pr\left(s_1^b|b\right)}\frac{\Pr\left(\hat{S}_2|a\right)}{\Pr\left(\hat{S}_2|b\right)}\frac{\Pr(s_2'|a)}{\Pr(s_2'|b)}\frac{\Pr\left(\hat{S}_1|a\right)}{\Pr\left(\hat{S}_1|b\right)},$$

or equivalently,

$$\frac{\Pr\left(\{s_1^a\} \times \hat{S}_2|a\right)}{\Pr\left(\{s_1^a\} \times \hat{S}_2|b\right)}\frac{\Pr\left(\hat{S}_1 \times \{s_2\}|a\right)}{\Pr\left(\hat{S}_1 \times \{s_2\}|b\right)} \geq \frac{\Pr\left(\left\{s_1^b\right\} \times \hat{S}_2|a\right)}{\Pr\left(\left\{s_1^b\right\} \times \hat{S}_2|b\right)}\frac{\Pr\left(\hat{S}_1 \times \{s_2'\}|a\right)}{\Pr\left(\hat{S}_1 \times \{s_2'\}|b\right)}.$$

It follows that at least one of the following pair of inequalities holds:

$$\frac{\Pr\left(\{s_1^a\} \times \hat{S}_2|a\right)}{\Pr\left(\{s_1^a\} \times \hat{S}_2|b\right)} \geq \frac{\Pr\left(\hat{S}_1 \times \{s_2'\}|a\right)}{\Pr\left(\hat{S}_1 \times \{s_2'\}|b\right)}$$

$$\frac{\Pr\left(\hat{S}_1 \times \{s_2\}|a\right)}{\Pr\left(\hat{S}_1 \times \{s_2\}|b\right)} \geq \frac{\Pr\left(\left\{s_1^b\right\} \times \hat{S}_2|a\right)}{\Pr\left(\left\{s_1^b\right\} \times \hat{S}_2|b\right)}.$$

This completes the proof of Lemma 4. ∎

We are now ready to establish Lemma 3. Let s_2^{Ta} and s_2^{Tb} respectively be the most pro-a and pro-b signals in S_2^T. By Lemma 4, either (i) $\{s_1^a\} \times S_2^T$ is more pro-a than $S_1 \times \left\{s_2^{Ta}\right\}$, or (ii) $\left\{s_1^b\right\} \times S_2^T$ is more pro-b than $S_1 \times \left\{s_2^{Tb}\right\}$. We will establish the claim for case (i). (Case (ii) follows symmetrically.)

Consider an element Q' of the seller's information partition $\mathcal{P}_1$ of the form $Q' = \{a,b\} \times \left\{s_1^b\right\} \times \hat{S}_2$, where $\hat{S}_2 \subset S_2^T$. (The fact that $\{a,b\} \times S_1 \times S_2^T$ is measurable with respect to $\mathcal{P}_1$ ensures that such an element exists.[38]) Expanding,

$$\begin{aligned}\Pr(a|Q') &= \frac{\Pr(a)\Pr(Q'|a)}{\Pr(Q')} = \frac{\Pr(a)\Pr\left(s_1^b|a\right)\Pr\left(\hat{S}_2|a\right)}{\Pr(Q')} \\ &= \Pr\left(s_1^b|a\right)\sum_{s_2\in\hat{S}_2}\frac{\Pr(a)\Pr(s_2|a)}{\Pr(Q')} = \Pr\left(s_1^b|a\right)\sum_{s_2\in\hat{S}_2}\frac{\Pr(s_2)\Pr(a|s_2)}{\Pr(Q')}.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\Pr(Q')}{\Pr\left(s_1^b|a\right)} &= \sum_{s_2\in\hat{S}_2}\left(\Pr(a)\frac{\Pr\left(s_1^b|a\right)}{\Pr\left(s_1^b|a\right)}\Pr(s_2|a) + \Pr(a)\frac{\Pr\left(s_1^b|b\right)}{\Pr\left(s_1^b|a\right)}\Pr(s_2|b)\right) \\ &> \sum_{s_2\in\hat{S}_2}\left(\Pr(a)\Pr(s_2|a) + \Pr(b)\Pr(s_2|b)\right) = \sum_{s_2\in\hat{S}_2}\Pr(s_2),\end{aligned}$$

where the inequality follows since the seller's signal is at least somewhat informative and so $\Pr\left(s_1^b|a\right) < \Pr\left(s_1^b|b\right)$. Thus we can write $\Pr(a|Q')$ in form

$$\Pr(a|Q') = \sum_{s_2\in\hat{S}_2} w(s_2)\Pr(a|s_2),$$

where $\left\{w(s_2) : s_2 \in \hat{S}_2\right\}$ is a set of weights summing to strictly less than unity. Consequently, there exists $\hat{s}_2 \in \hat{S}_2 \subset S_2^T$ such that

$$\Pr(a|Q') < \Pr(a|\hat{s}_2) = \Pr(a|S_1 \times \{\hat{s}_2\}).$$

Clearly $S_1 \times \left\{s_2^{Ta}\right\}$ is at least as pro-a as $S_1 \times \{\hat{s}_2\}$. Recall, moreover, that $\{s_1^a\} \times S_2^T$ is more pro-a than $S_1 \times \left\{s_2^{Ta}\right\}$ (we are in case (i)). So

$$\Pr\left(a|\{s_1^a\} \times S_2^T\right) \geq \Pr(a|S_1 \times \{\hat{s}_2\}) > \Pr(a|Q').$$

To complete the proof, simply observe that that the most pro-a element of the seller's information partition lying in $S_1 \times S_2^T$ is *at least* as pro-a as $\{s_1^a\} \times S_2^T$. ∎

[38] Moreover, agent i's information $\mathcal{P}_i$ is at least as fine as $\hat{\mathcal{P}}_i$