Due May 14 in class

1. vdV 11.1
2. vdV 11.5
3. vdV 11.6
4. vdV 11.8
5. Consider an estimating function $\psi_{\theta}(X)$ where the conditions of Theorem 5.41 are satisfied. Suppose that $\theta = (\beta, \alpha)$ and that for every fixed $\alpha$, $\beta \mapsto \psi_{\beta,\alpha}$ satisfies the conditions of theorem 5.41 (with the same true value $\beta_0$ for all $\alpha$)

Let $\hat{\alpha}_n$ be any sequence such that for some $\alpha^*$

$$\hat{\alpha}_n - \alpha^* = O_p(n^{-1/2}).$$

Define $\hat{\beta}_n$ to be the solution of

$$\frac{1}{n} \sum_i \psi_{\beta,\hat{\alpha}_n}(X_i) = 0$$

and show that $\hat{\beta}_n$ is consistent for $\beta_0$ and that its limiting distribution depends on $\hat{\alpha}_n$ only through $\alpha^*$. Can you show that

$$\hat{\alpha}_n - \alpha^* = o_p(1)$$

is sufficient?

[Note: In regression problems where the primary scientific interest is in parameters $\beta$ describing $E[Y|X]$ it is often possible to arrange for parameters $\alpha$ describing the distribution of $Y - E[Y|X]$ to satisfy these conditions. Within the exponential family there are even likelihood equations satisfying these conditions.]
Recurring problems for the project:

1. vdV 5.1

2. vdV 5.9

3. Suppose \((X, Y) \in \mathbb{R}^2\) are sampled from the bivariate distribution \(F\) with marginal distributions \(F_X\) and \(F_Y\). Consider the correlation functional

\[
\rho = \frac{\int xy - \mu_x \mu_y \, dF(x, y)}{\sqrt{\int (x - \mu_x)^2 \, dF_X(x) \int (y - \mu_y)^2 \, dF_Y(y)}}
\]

where \(\mu_x\) and \(\mu_y\) are the means of \(X\) and \(Y\).

The correlation is defined on the set of distributions with finite first and second moments. Show that it is not weakly continuous everywhere on this set and find a class of distributions where it is weakly continuous. Obviously there is more than one such class, and bigger classes are better.

4. Bayesian robustness: what happens when the likelihood conflicts with the prior

   (a) Suppose your prior for \(\mu\) is Cauchy with median zero, your data model is \(N(\mu, 1)\), and you observe one point \(X\). What happens to the posterior mode as \(X \to \infty\).

   (b) Suppose your prior for \(\mu\) is \(N(0, 1)\), your data model is Cauchy with location \(\mu\), and you observe one point \(X\). What happens to the posterior mode as \(X \to \infty\).