Supplementary Problems

10.18. Repeat Problem 10.3, using the raising and upper operators, \( a \) and \( a' \), respectively.

10.19. Consider a one-dimensional oscillator with a linear perturbation \( \lambda x \). For the ground state energy, compute the first two orders of the perturbation (use dimensionless units). Ans. \( E(\lambda) = 1 - \lambda^2 / 4 \).

10.20. A small perturbation, \( W = ax^4 \), is applied to a harmonic oscillator with force constant \( k \) and reduced mass \( m \). Compute the first-order correction to the eigenergies and first nonvanishing correction to the wave functions.

\[
E_s^{(1)} = \frac{3a}{2\alpha^2} \left(n^2 + n + \frac{1}{2}\right); \quad \phi_0 = \phi_0^{(0)} - \frac{3\sqrt{2}a}{4\hbar\omega\alpha^2} \phi_1^{(0)} + \cdots.
\]

10.21. Consider the Hamiltonian \( H = H_0 + V(x, y) \), where \( H_0 = m\omega^2 (x^2 + y^2) / 2 \) is the free Hamiltonian, where \( V(x, y) = \lambda m\omega^2 xy \) is a perturbation. (a) Find the exact ground state. (b) Using the second-order perturbation, calculate the ground state energy.

\[
\psi_0(x, y) = \frac{m\omega}{\sqrt{\pi\hbar}} (1 - \lambda)^{1/4} \exp \left( -\frac{m\omega}{\hbar} \left( \sqrt{1 - \lambda} + \sqrt{1 + \lambda} \right) x^2 \right)

\times \exp \left( -\frac{m\omega}{\hbar} \left( \sqrt{1 - \lambda} + \sqrt{1 + \lambda} \right) y^2 \right) \exp \left( -\frac{m\omega}{\hbar} \left( \sqrt{1 - \lambda} - \sqrt{1 + \lambda} \right) xy \right).
\]

\( E_0^{(2)} = -\frac{\lambda^2 \hbar \omega}{8} \), where the exact result is \( E_0 = \frac{\hbar \omega}{2} (\sqrt{1 - \lambda} + \sqrt{1 + \lambda}) \equiv \hbar \omega - \frac{\lambda^2 \hbar \omega}{8} + \cdots \).

10.22. A particle with mass \( m \) and electric charge \( e \) moves in a one-dimensional harmonic potential, subjected to a weak electric field \( \varepsilon \). (a) Calculate the corrections to the energy levels and to the eigenstates of the first nonvanishing order. (b) Calculate the electric dipole moment of the particle. (c) Solve parts (a) and (b) exactly, and compare the results to the approximate solutions.

\[
\Delta E = \frac{(\varepsilon e)^2}{2m\omega^2}; \quad |\psi_\varepsilon\rangle = |\phi_0\rangle - e\hbar \sum_{n=1}^{\infty} \left( \frac{\hbar}{m\omega^2} \right)^n \left( \sum_{j=-n}^{n} \sqrt{\binom{n}{j}} |\phi_{n-j}\rangle - \sqrt{\binom{n}{j}} |\phi_{n+j}\rangle \right).
\]

\( P = \frac{e\varepsilon e^2}{2m\omega^2} \).

10.23. A plane rotator with electric dipole moment \( \mathbf{d} \) and an inertia moment \( I \) is subject to a uniform electric field \( \mathbf{E} \) that lies in the plane of rotation. Calculate the first nonvanishing corrections to the energy levels of the rotor. Consider the field \( \mathbf{E} \) as a small perturbation. Hint: The perturbation is \( W = -\mathbf{d} \cdot \mathbf{E} \).

\[
E_n = E_n^{(0)} + E_n^{(2)} = \frac{\hbar^2 n^2}{2I} + \frac{(I\cdot E)^2}{\hbar^2 (4m^2 - 1)}.
\]

10.24. Consider the Hamiltonian

\[
H = \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{p}'^2}{2m} + \frac{1}{2} m\omega_0^2 (x_1^2 + x_2^2) + V_{nl} (x_1 - x_2) \tag{10.24.1}
\]

where \( V_{nl} (x_1 - x_2) = \frac{1}{4} m\omega_0^2 (x_1 - x_2)^2 \). (a) Find the exact energy of this system. (b) Assuming that \( W = V_{nl} (x_1 - x_2) \), use the second-order perturbation to compute the energy of the ground state.

\[
E_n = \left( N + 1/2 \right) \hbar \omega_0 + \left( n + 1/2 \right) \hbar \sqrt{\omega_0^2 + \omega_1^2} \approx \hbar \omega_0 + \frac{\hbar \omega_1^2}{4 \hbar \omega_0} \frac{\hbar \omega_0^4}{16 \hbar \omega_0^4} + \cdots \tag{10.24.2}.
\]

10.25. In the first approximation, compute the energy of the ground state of a two-electron atom or ion having a nuclear charge \( Z \). Considering the interaction between the electrons as a small perturbation.

\[
E = E^{(0)} + E^{(1)} = -\left( \frac{Z^2 - 4}{8} \right) \frac{e^4}{\hbar^2}.
\]