Overview of Topics

1. Roots of Unity
2. Branch cuts (optional)
3. Analytic functions and the Cauchy-Riemann Conditions.
Complex numbers in science fiction:

Consider relativistic mass dilation for very fast particle with velocity $v$ and rest mass $M_0$ (mass if $v=0$):

$$M = \frac{M_0}{\sqrt{1 - v^2/c^2}} \quad \text{C = speed of light}\$$

Usually we argue that particles cannot go faster than the speed of light because they would have $\infty$ mass as they pass through $v = c$ (divide by zero).

However, what about particles that always travel faster than light?

If $v > c$ (always) then mass is imaginary!

These particles are called Tachyons, and it has been proposed (dubiously) that neutrinos may tachyons...

"Oh no! Tachyons are flooding the warp core!"

Power function \( z^a \) may be rewritten using "exp" and "log":
\[
z = e^{\log(z)} \implies z^a = (e^{\log(z)})^a = e^{a \log(z)}
\]

This formula, \( z^a = e^{a \log(z)} \), is valid (and useful) for any real or complex-valued power \( a \).

**Example:** Take a rational \( a = \frac{m}{n} \in \mathbb{Q} \subset \mathbb{R} \) with \( mn \in \mathbb{Z} \)

\[
z = e^{\frac{m}{n} \log(1 |z|^2 + i(\theta_p + 2\pi n k))}
\]

\[
= e^{e^{\frac{m}{n} \log(R)} e^{\frac{m}{n} i\theta_p} e^{\frac{m}{n} i 2\pi n k}}
\]

\[
\implies z = R e^{\frac{m}{n} i(\frac{m}{n})(\theta_p + 2\pi k)}
\]

This has \( n \) unique values for \( k = 0, 1, 2, \ldots, n-1 \)

i.e. \( \frac{m}{n} \left[ \theta_p + 2\pi n k \right] = \frac{m}{n} \theta_p + 2\pi m \), which is same as \( \frac{m}{n} \theta_p \) for \( k = 0 \).

**Example:** Roots of unity \( \sqrt[n]{1} = 1^{\frac{1}{n}} = e^{i\frac{2\pi k}{mn}} \) for \( k = 0, 1, 2, \ldots, n-1 \).

Three values for \( \sqrt[3]{1} = 1^{\frac{1}{3}} \)

Three values for \( \sqrt[3]{-1} = (-1)^{\frac{1}{3}} \)

also works for \( a \) irrational (oo-many values) or complex...
**Branch Points & Cuts (multivalued functions):**

A branch of a multivalued function $f(z)$ is a single-valued analytic function $F(z)$ on a region $\mathcal{D} \subseteq \mathbb{C}$ that coincides with $f(z)$ on one branch.

**Branch cut $B \subseteq \mathbb{C}$** is a curve that bounds the region $\mathcal{D}$. The points $z \in B$ are singular, meaning that $F(z)$ jumps values on either side of $B$.

A branch point is a point common to all branch cuts.

**Example:** $\text{Log}(z) = \log|z| + i\theta_p$ is the principal branch of $\text{Log}(z)$.

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**Diagram:**

- The principal branch of $\text{Log}(z)$.
- Branch cut and branch point.
- $\Theta$ jumps by $\pi i$ when branch cut is crossed.
Analytic Functions:

We would like to be able to do Calculus in the complex plane. However, some functions are better behaved than others.

A function is **analytic** in a domain $\Omega$ if $f(z)$ is single-valued and has a finite derivative $f'(z)$ for all $z \in \Omega$.

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Example of weird function that is not analytic:

$f(z) = \overline{z} := x - iy$.

$$
\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta \overline{z}}{\Delta z} = \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y}
$$

Let $\Delta z = \Delta x + i \Delta y$ so $\overline{\Delta z} = \Delta x - i \Delta y$.

1. Approach $\Delta z = 0$ from real axis (i.e. $\Delta y = 0$): $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = +1$

2. Approach $\Delta z = 0$ from imaginary axis (i.e. $\Delta x = 0$): $\lim_{\Delta y \to 0} \frac{-i \Delta y}{i \Delta y} = -1$

{Different based on direction!!}

So $f'(z)$ is not a (single) finite value, and hence $f(z) = \overline{z}$ is not analytic.
At the least, for a function to be analytic, the derivative must be the same from the two paths taken on the real and imaginary axes.

\[ f(z) = u(x,y) + iv(x,y) \quad \text{where } x,y \text{ are from } z = x + iy \]

So \[ \frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} \]

1. **Approach on Real axis (\( \Delta y = 0 \))**: \[ \frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u + i \Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \]

2. **Approach on Imaginary axis (\( \Delta x = 0 \))**: \[ \frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta u + i \Delta v}{i \Delta y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \]

A necessary condition for \( \frac{df}{dz} \) to exist is for \( 1 = 2 \).

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

Cauchy–Riemann Conditions (CR)

It turns out that the CR conditions are both necessary and sufficient as long as all partials are continuous.