Quiz 4

Show all your work. No credit is given without reasonable supporting work. There are two sides to this quiz and all logic symbols make use of the textbook notation.

1. [2] The “structure” of induction requires two steps. Name the two steps.
    - Base Case: Show statement holds on initial step
    - Inductive Step: Show if statement holds on k^{\text{th}} step, it holds on the next

2. [3] (§5.1 Example 15) Determine if the following proof is valid or not. If valid, identify the proof method(s) used. If not valid, highlight the fallacies/error in logic.

**Theorem 1.** Every set of lines in the plane, no two of which are parallel, meet in a common point.

**Proof 1.** Certainly two lines that are not parallel meet in a common point.

Assume a collection of k lines, no two of which are parallel, meet at a common point. We will show that a collection of k + 1 lines, no two of which are parallel, will also meet at a common point.

Consider the set of k + 1 distinct lines in the plane. By our hypothesis, the first k of these lines meet in a common point p_1. Similarly the last k of these lines meet in a common point p_2. We will show that p_1 and p_2 must be the same point.

If p_1 and p_2 were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, p_1 and p_2 are the same point. We conclude that the point p_1 = p_2 lies on all k + 1 lines, thus our k + 1 distinct lines meet at a common point.

3. [3] Explain how inductions works as you would to a colleague who “doesn’t believe” in induction.
4. Assume that a chocolate bar consists of \( n \) squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical of a horizontal line separating the squares. Assume that only one piece can be broken at a time. Prove you will need \( n - 1 \) breaks to break the bar into \( n \) separate squares.

We will use (stronger) induction on the number of squares in the chocolate bar.

**Base Case:** If \( n = 1 \), there are no breaks or 0 breaks needed.

**Induction:** Assume a chocolate bar with \( k \) squares takes \( k - 1 \) breaks for \( k \).

We want to show a bar with \( k + 1 \) squares takes \( k \) breaks.

Case 1: The bar with \( k + 1 \) squares has only 1 row. Clearly, the bar will take \( k \) (the # of dividers between the squares).

Case 2: The bar with \( k + 1 \) squares has more than 1 row. Break the original bar along any row dividing the bar into two smaller bars with \( m \) and \( k+1-m \) squares. Note \( m \leq k \) and \( k+1-m \leq k \). By the inductive assumption, the two smaller bars take \( m-1 \) and \( k+1-m-1 \) breaks respectively. Thus the total # of breaks for our \( k + 1 \) squares bar is \( 3k + 1 \).

5. Prove that \( 3 + 3 \cdot 5 + 3 \cdot 5^2 + 3 \cdot 5^3 + \cdots + 3 \cdot 5^n = \frac{3(5^{n+1} - 1)}{4} \) whenever \( n \) is a nonnegative integer.

We will prove this statement with induction:

**Base Case:** Let \( n = 0 \) then \( 3 \cdot 5^0 = \frac{3}{4} = \frac{3(5^1 - 1)}{4} \).

**Induction:** Assume \( \sum_{i=0}^{k} 3 \cdot 5^i = \frac{3(5^{k+1} - 1)}{4} \).

We want to show \( \sum_{i=0}^{k+1} 3 \cdot 5^i = \frac{3(5^{k+1+1} - 1)}{4} \).

Consider \( \sum_{i=0}^{k+1} 3 \cdot 5^i = 3 \cdot 5^0 + 3 \cdot 5^1 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} \)

We add \( 3 \cdot 5^{k+1} \) to both sides to get

\[
\sum_{i=0}^{k+1} 3 \cdot 5^i = \frac{3(5^{k+1} - 1)}{4} + 3 \cdot 5^{k+1}
\]

By the inductive assumption, with algebra

\[
= 3 \left[ \frac{5^{k+1} - 1}{4} + \frac{5 \cdot 5^{k+1}}{4} \right]
\]

\[
= 3 \left[ \frac{5^{k+1} + 5 \cdot 5^{k+1} - 1}{4} \right] = 3 \left[ \frac{6 \cdot 5^{k+1} - 1}{4} \right] = \frac{3(5^{k+2} - 1)}{4}
\]