Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [ ] TRUE/FALSE: Circle T in each of the following cases if the statement is always true. Otherwise, circle F. Let $f$ and $g$ be differentiable functions and $h$ be a constant.

   T  \[ \frac{x+h}{2x} = \frac{1+h}{x} \]

   $\frac{h}{x} = \frac{2(\ln x)}{2x} = \frac{2 \ln x}{2x}$

   T  \[ \sqrt{x^2 + h^2} = x + h \]

   Let $x = 1$ and $h = 1$ we see $\sqrt{1 + 1} \neq 1 + 1$

   T  \[ \lim_{x \to r} f(x) = f(r) \] for all $r$ in the domain of $f$.

   T  \[ \lim_{x \to r} g(x) = 0, \text{ then } \lim_{x \to r} \frac{f(x)}{g(x)} \text{ does not exist.} \]

   T  \[ \frac{d}{dx} \left( \frac{1}{x^2} \right) = -1 \]

   Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [ ] Sketch the graph and then find the formula of an example function $f$ that satisfies the following conditions:

   (a) $f(2) = 2$
   (b) $\lim_{x \to 2^-} f(x) = -4$
   (c) $f$ is not differentiable when $x = -3$
   (d) $f$ is continuous when $x = -3$
   (e) $\lim_{x \to -3^+} f(x) = \infty$
   (f) $f'(4) = 2$
3. Compute the following limits:

(a) \[ \lim_{x \to 1} \frac{x^2 + x - 2}{2x^2 - 8x + 6} = \lim_{x \to 1} \frac{(x+2)(x-1)}{2(x-3)(x-1)} = \lim_{x \to 1} \frac{x+2}{2(x-3)} = \frac{3}{-4} \]

(b) \[ \lim_{x \to \infty} \frac{x^2 + x - 2}{2x^2 - 8x + 6} = \lim_{x \to \infty} \frac{1 + \frac{1}{x} - \frac{2}{x^2}}{2 - \frac{8}{x} + \frac{6}{x^2}} = \frac{\lim_{x \to \infty} \left(1 + \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty} \left(2 - \frac{8}{x} + \frac{6}{x^2}\right)} = \frac{1}{2} \]

(c) \[ \lim_{\theta \to 0^+} \frac{\theta + \theta^2}{1 - \cos \theta} = \lim_{\theta \to 0^+} \frac{1 + \frac{1}{\theta}}{1 + \sin \theta} \]

(d) \[ \lim_{x \to 0^+} x \sin \left(\frac{5\pi}{x}\right) = \lim_{x \to 0^+} \frac{x \sin \left(\frac{5\pi}{x}\right)}{\frac{5\pi}{x}} = \lim_{x \to 0^+} \cos \left(\frac{5\pi}{x}\right) = \lim_{x \to \infty} \frac{5\pi}{x} \cos \left(\frac{5\pi}{x}\right) = \frac{5\pi}{x} \lim_{x \to \infty} \cos \left(\frac{5\pi}{x}\right) = 5\pi \frac{1}{x} = \frac{5\pi}{x} \]

(e) \[ \lim_{x \to 0} x^4 \sin \left(\frac{1}{x}\right) \]

Note: \(\sin \frac{1}{x}\) as \(x \to 0\) never 'settles down' but for all \(x\)
\[-1 \leq \sin \left(\frac{1}{x}\right) \leq 1\]

\[\Rightarrow \text{ if we multiply the inequalities by } x^4 \]
\[-x^4 \leq x^4 \sin \left(\frac{1}{x}\right) \leq x^4\]

Observe \(\lim_{x \to 0^+} x^4 = 0 = \lim_{x \to 0^-} x^4\)

So by the Squeeze Theorem
\[\lim_{x \to 0} x^4 \sin \left(\frac{1}{x}\right) = 0.\]

(f) \[ \lim_{x \to 1^-} \frac{1}{x-1} \]

Note \(\frac{1}{x-1}\) looks like the graph of \(\frac{1}{x}\) shifted one unit to the right 1 unit

\[\Rightarrow \lim_{x \to 1^-} \frac{1}{x-1} = -\infty\]

Thus \(\lim_{x \to 1^-} \frac{1}{x-1}\) doesn't exist.
4. Let \( f(x) = \begin{cases} \sqrt{1-(x+3)^2} & \text{if } -4 \leq x \leq -2 \\ 1 & \text{if } -2 < x < 1 \\ -(x-2)^2 + 2 & \text{if } 1 < x \end{cases} \)

Graph \( f(x) \) and then sketch the graph \( f'(x) \) below on its own set of axes. Afterwards, answer the following questions.

(a) \( \lim_{x \to 1} f(x) \)

(b) \( \lim_{x \to -3} [4f(x) - 7] \)

(c) \( \lim_{x \to 2} f(x) \)

(d) \( \lim_{x \to 2} f(x) \)

(e) \( \lim_{x \to 3} f'(x) \)

(f) \( \lim_{x \to \infty} f(x) \)

(g) \([f + f]'(2) = (f(2) + f'(2))\)

\[= 0 + 0 = 0\]

\[\text{Note: } y^2(x+3)^2 = 1 \]
\[\frac{d}{dx}(y^2(x+3)^2) = \frac{d}{dx}(1) \]
\[\frac{d}{dx}y^2 + 2(x+3)y\frac{dy}{dx} = 0 \]
\[y' = -\frac{\frac{d}{dx}(x+3)}{\frac{dy}{dx}} \]
\[y' = -\frac{(x+3)}{1-(x+3)} \]
5. Compute the derivatives of the following functions. You do not need to simplify.

(a) \( f(x) = x^3 + 3^x + \pi^x \)

\[ f'(x) = 3x^2 + 3^x \ln(3) \]

(b) \( g(t) = \ln(t) \left( \frac{2 + t^2}{3t - 1} \right) \)

\[ g'(t) = \ln(t) \left[ \frac{2 + t^2}{3t - 1} \right]' + \frac{\ln(t)}{3t - 1} \left( \frac{2 + t^2}{3t - 1} \right) \]

\[ = \ln(t) \left[ \frac{(3t - 1)(2t) - (2 + t^2)(3)}{(3t - 1)^2} \right] + \frac{1}{t} \left( \frac{2 + t^2}{3t - 1} \right) \]

(c) \( h(\theta) = 7 \sec(\sqrt{\theta}) \)

\[ h'(\theta) = 7 \left[ \sec(\sqrt{\theta}) \right]' \]

\[ = 7 \cdot -1 \left( \cos(\sqrt{\theta}) \right)^{-2} \cdot (-\sin(\sqrt{\theta})) \frac{1}{2} \theta^{-\frac{1}{2}} \]

\[ = -\frac{7 \sin(\sqrt{\theta})}{\theta} \cdot \frac{1}{2 \sqrt{\theta}} \]

\[ = -\frac{7 \sin(\sqrt{\theta})}{\theta (\cos(\sqrt{\theta}))^2} \]

(d) \( y = \sqrt{x e^{3x}}(x^6 + 3)^{10} \)

\[ y' = \frac{1}{\sqrt{x e^{3x}}} \cdot \frac{1}{2} \cdot \frac{1}{x} + 7x^6 + \frac{10}{x^6 + 3} \cdot \frac{d}{dx} \left( x^6 + 3 \right) \]

\[ = \frac{1}{\sqrt{x e^{3x}}} \cdot \frac{1}{x} + 7x^6 + \frac{10}{x^6 + 3} \cdot 6x^5 \]

\[ y' = \frac{1}{\sqrt{x e^{3x}}} \cdot \frac{1}{x} + 7x^6 + \frac{60x^5}{x^6 + 3} \]

(e) \( y = \left( \cos(x) \right)^x \)

\[ \frac{dy}{dx} = \ln(\cos(x)) \cdot x \]

\[ \frac{dy}{dx} = x \ln(\cos(x)) \]

\[ \frac{dy}{dx} = x \ln(\cos(x)) \]

\[ \frac{dy}{dx} = x \left[ x \tan(x) + \ln(\cos(x)) \right] \]

\[ y' = y \left[ x \tan(x) + \ln(\cos(x)) \right] \]

(d) \( x^2 y^2 = 4 - y \arctan(5x) \)

\[ x^2 y x^2 y' + 2xy^2 = \frac{1}{1 + 125x^2} \cdot 5 + y' \arctan(5x) \]

\[ 2x^2 y y' + 2xy^2 = \frac{-5y}{1 + 125x^2} - y' \arctan(5x) \]

\[ 2x^2 y y' + y' \arctan(5x) = 5x \]

\[ y' \left( 2x^2 y + \arctan(5x) \right) = 5x \]

\[ y' = \frac{-5x}{1 + 125x^2 - 2xy^2} \]

\[ \frac{dy}{dx} = \frac{-5x}{1 + 125x^2 - 2xy^2} \]
6. Find the equation of the line tangent to the graph of \( f \) when \( x = 2 \) if \( f(x) = m(n(x)) \), \( n(2) = -1 \), \( m(-1) = 6 \), \( n'(2) = 3 \), and \( m'(-1) = 5 \).

\[
\begin{align*}
\text{Looking for } y &= mx + b \\
\text{So we have } y &= 15x + b \\
m &= f'(2) \\
f'(x) &= (m \circ n)'(x) \text{ Chain Rule} \\
 &= m'(n(x)) \cdot n'(x) \\
So \quad f'(2) &= m'(n(2)) \cdot n'(2) \\
 &= m'(-1) \cdot 3 = 5.3 = 15
\end{align*}
\]

Thus \( y = 15x - 34 \)

7. Find the antiderivative for each of the following functions:

(a) \( 2x - x^3 + 7\sin(x) \)

\[
x^2 - \frac{1}{4}x^4 + 7\cos(x)
\]

Check:

\[
(x^2 - \frac{1}{4}x^4 + 7\cos(x))' = 2x - x^3 + 7(-\sin(x))
\]

off by negative sign in last term so

\[
x^2 - \frac{1}{4}x^4 - 7\cos(x)
\]

(b) \( \frac{5 - 4x^3 + 2x^6}{x^6} \)

\[
= \frac{5}{x^6} - \frac{4x^3}{x^6} + \frac{2x^6}{x^6}
\]

\[
= 5x^{-6} - 4x^{-3} + 2.
\]

8. Consider the function \( f(x) = \sqrt{x} \)

(a) Evaluate the integral \( \int_1^8 \sqrt{x} \, dx = F(8) - F(1) \) where \( F \) is an antiderivative

\[
(\frac{3}{4} \cdot \frac{4}{5})^{1/2} = \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{1}{2} = \frac{3}{10} \\
So \quad \frac{3}{4} \cdot \sqrt{\frac{4}{5}} \text{ is an antiderivative} \int_1^8 \sqrt{x} \, dx = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{10}
\]

(b) Draw a picture that corresponds to the area you computed in (a).
9. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of 2m³/min, find the rate at which the water level is rising when the water is 3m deep.

Let \( V \) be the volume of the water. Recall the volume of a cone is \( \frac{1}{3} \pi r^2 h \), where \( r \) is the radius and \( h \) is the height.

\[
V = \frac{\pi}{3} r^2 h
\]

It would be easier to take the derivative if we had only 1 variable. The relation between \( r \) and \( h \) is:

\[
\frac{r}{h} = \frac{2}{h} \Rightarrow r = \frac{2h}{3}
\]

So \( V = \frac{\pi}{3} \left( \frac{2h}{3} \right)^2 h \) when \( h = 3 \),

\[
\frac{dV}{dt} = \frac{\pi}{27} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\frac{\pi}{27} h^2}{\frac{dV}{dt}} = \frac{\pi}{27} \cdot \frac{3^2}{2} \cdot \frac{1}{2} = \frac{8}{9 \pi}
\]

10. When blood flows along a blood vessel, the flux \( F \) (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius \( R \) of the blood vessel: \( F = kR^4 \). A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Use a linear approximation to show that the relative change in \( F \) is about four times the relative change in \( R \). Then approximate how a 5% increase in the radius will affect the flow of blood.

Let \( a \) be the radius of a blood vessel,

we want to find \[
\frac{\text{error in } F}{\text{at } a} = \frac{\Delta F}{F(a)}
\]

which is well approximated by

\[
\frac{\Delta F}{F(a)} \approx \frac{4}{a} \frac{\Delta a}{a}
\]

so \( \frac{\Delta F}{F(a)} = 4\frac{\Delta a}{a} \)

\[
4\frac{a^3}{\Delta a} = \frac{\Delta F}{F(a)}
\]

\[
\Rightarrow \Delta F = 4\frac{a^3}{\Delta a}
\]

\[
\Rightarrow \Delta F(a) = 4\frac{a^3 \Delta R}{\Delta a} = 4\frac{\Delta R}{a}
\]

Note \( F = 4\frac{\Delta R}{a} \).
11. Find the dimensions of the rectangle of largest area that has its base on the $x$-axis and its other two vertices above the $x$-axis and lying on the parabola $y = 7 - x^2$.

\[
\text{Area} = \text{length} \times \text{height} \\
\text{note } w = 2 \cdot x \\
\text{and } h = y = 7 - x^2 \\
\text{so } \text{Area} = 2x(7-x^2) = 14x - 2x^3 \\
\text{To maximize we need to find the extrema...} \\
\text{Area}' = 14 - 6x^2 \\
0 = 14 - 6x^2 \\
-14 = -6x^2 \\
x = \pm \sqrt{\frac{14}{6}} = \pm \sqrt{\frac{7}{3}} \\
\text{so width} = 2\sqrt{\frac{7}{3}} \\
\text{and height} = 7 - \sqrt{\frac{7}{3}} = \frac{14}{3}
\]

12. A truck has a minimum speed of 9 mph in high gear. When traveling $x$ mph, the truck burns diesel fuel at the rate of

\[
0.003935 \left( \frac{675}{x} + x \right) \text{ gal/mile}
\]

Assume that the truck can not be driven over 63 mph, that diesel fuel costs $2.84 a gallon, and that the driver is paid $12 an hour. Find the speed that will minimize the cost of a 500 mile trip.

\[
\text{Total cost} = \text{Cost of Gas} + \text{Cost of driver} \\
\text{Cost of Gas:} \\
0.003935 \left( \frac{675}{x} + x \right) \text{ gal/mile} \times 0.500 \text{ mile} \times 2.84 \text{ dollars/gal} = 5.5877 \left( \frac{675}{x} + x \right) \text{ dollars} \\
\text{Cost of driver:} \\
12 \text{ \$/hr} \times \frac{x \text{ miles}}{x \text{ miles}} \times 0.500 \text{ mile} = \frac{6000}{x} \text{ dollars} \\
\]

Note: $41.8 \text{ mph}$ won't work, only $41.8 \text{ mph}$ is a minimum.

Max Profit $\rightarrow 42 \text{ mph}$.