Word Problem Practice

1. Test makers use item response functions $P(x)$ to determine the difficulty and effectiveness of a given test question. The variable $x$ is the ability of a test taker and $P(x)$ is the probability that the test taker gets the problem correct. By convention we let an “average ability” correspond with $x = 0$. Thus $P(0) = .75$ means that a person with average ability has a 75% chance of getting the question correct.

(a) Find $\lim_{x \to \infty} P(x)$ and explain its meaning.

(b) Assume the question is a True/False question, find $\lim_{x \to -\infty} P(x)$. Justify yourself.

(c) Assume the question is multiple choice with 4 answers, find $\lim_{x \to -\infty} P(x)$.

2. In algorithms class you are given an assignment to sort $n$ integers within a given range. You would like to use bubble sort or bucket sort. The bubble sort algorithm’s efficiency is dependent on $n$ and in a worse case scenario could take $n^2$ steps. Computer scientists would say, that bubble sort is “big O” of $n^2$. By contrast, bucket sort is big O of $n$. Generally to determine if one algorithm is better than another, computer scientists consider the limit as $n$ goes to infinity of the ratio of the two big O’s. Perform this operation and compute $\lim_{n \to \infty} \frac{n}{n^2}$ and use the answer to determine which algorithm is better.

3. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

expresses the length $L$ of an object as a function of its velocity $v$ with respect to an observer, where $L_0$ is the length of the object at rest and $c$ is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result as a physicist.

4. Explain how scientists know there are at least two points directly opposite each other on the surface of the earth that are the same temperature.

5. A tank contains 8000L of pure water. Brine that contains 25 g of salt per liter of water is pumped into the tank at a rate of 25L/min. Find a function that records the concentration of salt after $t$ minutes (in grams per liter) and then find out what happens to the concentration as $t \to \infty$.

6. Consider a model of motion first advanced by Galileo. Galileo’s Law states that the distance a freely falling object falls is proportional to the square of the time it has fallen. Let us take $s(t)$ to be the distance an object falls in $t$ seconds from the top of a building 100 meters in height. The proportionality constant in Galileo’s Law can be determined, by experiment, to be 4.9 near the Earth’s surface. We thus have that $s(t) = 4.9t^2$. Find the velocity of the object after 3 seconds.
7. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height is meters \( t \) seconds later is given by \( y = 10t - 1.86t^2 \). Find the instantaneous velocity of the rock after one second. Find when the rock hits the max height.

8. Recall Newton’s Law of Cooling: If \( D_0 \) is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature \( T_s \), then the temperature of the objects at time \( t \) is modeled by the function

\[
T(t) = T_s + D_0 e^{-kt}
\]

where \( k \) is a positive constant that depends on the type of object. Find \( \lim_{t \to \infty} T(t) \) and interpret the result as a scientist.

9. A logistic growth model is a function that often describes the size of a population at time \( t \) and is of the form

\[
f(t) = \frac{c}{1 + ae^{-bt}}
\]

where \( a, b, \) and \( c \) are positive constants specific to the population under study. Find \( \lim_{t \to \infty} f(t) \) and interpret the result as a biologist.

10. The figure below shows a fixed circle \( C_1 \) with equation \((x - 1)^2 + y^2 = 1\) and a shrinking circle \( C_2 \) with radius \( r \) and center the origin. \( P \) is the point \((0, r)\), \( Q \) is the upper point of intersection of the two circles, and \( R \) is the point of intersection of the line \( PQ \) and the \( x \)-axis. What happens to \( R \) as \( C_2 \) shrinks, that is, as \( r \to 0^+ \)?