Outline

- Scalar nonlinear conservation laws
- Shocks and rarefaction waves
- Entropy conditions
- Finite volume methods
- Approximate Riemann solvers
- Lax-Wendroff Theorem

Reading: Chapter 11, 12

Burgers’ equation

Quasi-linear form: \( u_t + uu_x = 0 \)

The solution is constant on characteristics so each value advects at constant speed equal to the value...

Equal-area rule:

The area “under” the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.
Riemann problem for Burgers’ equation

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad u_t + u u_x = 0. \]

\[ f(u) = \frac{1}{2} u^2, \quad f'(u) = u. \]

Consider Riemann problem with states \( u_\ell \) and \( u_r \).

For any \( u_\ell, u_r \), there is a weak solution consisting of this discontinuity propagating at speed given by the Rankine-Hugoniot jump condition:

\[ s = \frac{\frac{1}{2} u_r^2 - \frac{1}{2} u_\ell^2}{u_r - u_\ell} = \frac{1}{2}(u_\ell + u_r). \]

Note: Shock speed is average of characteristic speed on each side.

This might not be the physically correct weak solution!

Burgers’ equation

The solution is constant on characteristics so each value advects at constant speed equal to the value...

Weak solutions to Burgers’ equation

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad u_\ell = 1, \quad u_r = 2 \]

Characteristic speed: \( u \)  \quad Rankine-Hugoniot speed: \( \frac{1}{2}(u_\ell + u_r) \).

“Physically correct” rarefaction wave solution:
Weak solutions to Burgers’ equation

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad u_\ell = 1, \quad u_r = 2 \]

Characteristic speed: \( u \)  
Rankine-Hugoniot speed: \( \frac{1}{2} (u_\ell + u_r) \).

Entropy violating weak solution:

Vanishing viscosity solution

We want \( q(x,t) \) to be the limit as \( \epsilon \to 0 \) of solution to

\[ q_t + f(q)_x = \epsilon q_{xx}. \]

This selects a unique weak solution:

- Shock if \( f'(q_l) > f'(q_r) \),
- Rarefaction if \( f'(q_l) < f'(q_r) \).

Lax Entropy Condition:

A discontinuity propagating with speed \( s \) in the solution of a convex scalar conservation law is admissible only if

\[ f'(q_l) > s > f'(q_r), \text{ where } s = (f(q_r) - f(q_l))/(q_r - q_l). \]

Note: This means characteristics must approach shock from both sides as \( t \) advances, not move away from shock!
Riemann problem for scalar nonlinear problem

\[ q_t + f(q)_x = 0 \]
with data

\[ q(x, 0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases} \]

Piecewise constant with a single jump discontinuity.

For Burgers’ or traffic flow with quadratic flux, the Riemann solution consists of:

- Shock wave if \( f'(q_l) > f'(q_r) \),
- Rarefaction wave if \( f'(q_l) < f'(q_r) \).

Five possible cases:

Transonic rarefactions

Sonic point: \( u_s = 0 \) for Burgers’ since \( f'(0) = 0 \).

Consider Riemann problem data \( u_\ell = -0.5 < 0 < u_r = 1.5 \).

In this case wave should spread in both directions:

Entropy-violating approximate Riemann solution:

\[ s = \frac{1}{2}(u_\ell + u_r) = 0.5. \]

Wave goes only to right, no update to cell average on left.
Transonic rarefactions

If \( u_\ell = -u_r \) then Rankine-Hugoniot speed is 0:

Similar solution will be observed with Godunov's method if entropy-violating approximate Riemann solver used.

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Notes:

Entropy-violating numerical solutions

Riemann problem for Burgers' equation at \( t = 1 \)
with \( u_\ell = -1 \) and \( u_r = 2 \):

Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive.

Often the nonlinear problem \( q_t + f(q)_x = 0 \) is approximated by

\[
q_t + A_{i-1/2} q_x = 0, \quad q_\ell = Q_{i-1}, \quad q_r = Q_i
\]

for some choice of \( A_{i-1/2} \approx f'(q) \) based on data \( Q_{i-1}, Q_i \).

Solve linear system for \( \alpha_{i-1/2} : Q_i - Q_{i-1} = \sum_p \alpha_{i-1/2}^p s_{i-1/2}^p \).

Waves \( \lambda_{i-1/2}^p = \alpha_{i-1/2}^p s_{i-1/2}^p \) propagate with speeds \( s_{i-1/2}^p \).

\( r_{i-1/2}^p \) are eigenvectors of \( A_{i-1/2} \),
\( s_{i-1/2}^p \) are eigenvalues of \( A_{i-1/2} \).
Approximate Riemann solvers

\[ q_t + \hat{A}_{i-1/2} q_x = 0, \quad q_t = Q_i-1, \quad q_r = Q_i \]

Often \( \hat{A}_{i-1/2} = f'(Q_{i-1/2}) \) for some choice of \( Q_{i-1/2} \).

In general \( \hat{A}_{i-1/2} = \hat{A}(q_t, q_r) \).

Roe conditions for consistency and conservation:
- \( \hat{A}(q_t, q_r) \rightarrow f'(q^*) \) as \( q_t, q_r \rightarrow q^* \),
- \( \hat{A} \) diagonalizable with real eigenvalues,
- For conservation in wave-propagation form,
  \[ \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}). \]

Notes:
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IPDE 2011, July 1, 2011 [FVMHP Sec. 15.3.2]

Approximate Riemann solvers

For a scalar problem, we can easily satisfy the Roe condition

\[ \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}). \]

by choosing

\[ \hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}. \]

Then \( r_{i-1/2}^t = 1 \) and \( s_{i-1/2}^t = \hat{A}_{i-1/2} \) (scalar!).

Note: This is the Rankine-Hugoniot shock speed.

\[ \longrightarrow \text{ shock waves are correct,} \]
\[ \text{rarefactions replaced by entropy-violating shocks.} \]

Notes:
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Approximate Riemann solver

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2} \right]. \]

For scalar advection \( m = 1 \), only one wave.
\( W_{i-1/2} = \Delta Q_{i-1/2} = Q_i - Q_{i-1} \) and \( s_{i-1/2} = u, \)

\[ A^- \Delta Q_{i-1/2} = s_{i-1/2}^- W_{i-1/2}, \]
\[ A^+ \Delta Q_{i-1/2} = s_{i-1/2}^+ W_{i-1/2}. \]

For scalar nonlinear: Use same formulas with \( W_{i-1/2} = \Delta Q_{1/2}, s_{i-1/2} = \Delta F_{i-1/2}, \Delta Q_{i-1/2} \).

Need to modify these by an entropy fix in the trans-sonic rarefaction case.

Notes:
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Entropy fix

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2} \right). \]

Revert to the formulas

\[ A^- \Delta Q_{i-1/2} = f(q_{i-1}) - f(q_{i}) \quad \text{left-going fluctuation} \]
\[ A^+ \Delta Q_{i+1/2} = f(q_{i}) - f(q_{i+1}) \quad \text{right-going fluctuation} \]

if \( f'(Q_{i-1}) < 0 < f'(Q_{i}). \)

High-resolution method: still define wave \( \mathcal{W} \) and speed \( s \) by

\[ \mathcal{W}_{i-1/2} = Q_i - Q_{i-1}, \]
\[ s_{i-1/2} = \begin{cases} f(Q_i) - f(Q_{i-1})/(Q_i - Q_{i-1}) & \text{if } Q_{i-1} \neq Q_i \\ f'(Q_i) & \text{if } Q_{i-1} = Q_i. \end{cases} \]

Godunov flux for scalar problem

The Godunov flux function for the case \( f''(q) > 0 \) is

\[ F_{i-1/2}^n = \begin{cases} f(Q_{i-1}) & \text{if } Q_{i-1} > q_s \text{ and } s > 0 \\ f(Q_i) & \text{if } Q_i < q_s \text{ and } s < 0 \\ f(q_s) & \text{if } Q_{i-1} < q_s < Q_i. \end{cases} \]

\[ = \begin{cases} \min_{Q_{i-1} \leq q \leq Q_i} f(q) & \text{if } Q_{i-1} \leq Q_i \\ \max_{Q_i \leq q \leq Q_{i-1}} f(q) & \text{if } Q_i \leq Q_{i-1}. \end{cases} \]

Here \( s = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}} \) is the Rankine-Hugoniot shock speed.

Entropy-violating numerical solutions

Riemann problem for Burgers’ equation at \( t = 1 \)

with \( u_L = -1 \) and \( u_R = 2 \):
Entropy (admissibility) conditions

We generally require additional conditions on a weak solution to a conservation law, to pick out the unique solution that is physically relevant.

In gas dynamics: entropy is constant along particle paths for smooth solutions, entropy can only increase as a particle goes through a shock.

Entropy functions: Function of \( q \) that “behaves like” physical entropy for the conservation law being studied.

NOTE: Mathematical entropy functions generally chosen to decrease for admissible solutions, increase for entropy-violating solutions.

Entropy functions

A scalar-valued function \( \eta : \mathbb{R}^m \rightarrow \mathbb{R} \) is a convex function of \( q \) if the Hessian matrix \( \eta''(q) \) with \((i,j)\) element

\[
\eta_{ij}''(q) = \frac{\partial^2 \eta}{\partial q_i \partial q_j}
\]

is positive definite for all \( q \), i.e., satisfies

\[
v^T \eta''(q) v > 0 \quad \text{for all } q, \ v \in \mathbb{R}^m.
\]

Scalar case: reduces to \( \eta''(q) > 0 \).

Entropy functions

Entropy function: \( \eta : \mathbb{R}^m \rightarrow \mathbb{R} \)  Entropy flux: \( \psi : \mathbb{R}^m \rightarrow \mathbb{R} \) chosen so that \( \eta(q) \) is convex and:

- \( \eta(q) \) is conserved wherever the solution is smooth,

\[
\eta(q)_t + \psi(q)_x = 0.
\]

- Entropy decreases across an admissible shock wave.

Weak form:

\[
\int_{x_1}^{x_2} \eta(q(x,t_2)) \, dx \leq \int_{x_1}^{x_2} \eta(q(x,t_1)) \, dx + \int_{t_1}^{t_2} \psi(q(x_1,t)) \, dt - \int_{t_1}^{t_2} \psi(q(x_2,t)) \, dt
\]

with equality where solution is smooth.
Entropy functions

How to find \( \eta \) and \( \psi \) satisfying this?

\[
\eta(q)_t + \psi(q)_x = 0
\]

For smooth solutions gives

\[
\eta'(q)q_t + \psi'(q)q_x = 0.
\]

Since \( q_t = -f'(q)q_x \) this is satisfied provided

\[
\psi'(q) = \eta'(q)f'(q).
\]

Scalar: Can choose any convex \( \eta(q) \) and integrate.

Example: Burgers’ equation, \( f'(u) = u \) and take \( \eta(u) = u^2 \).

Then \( \psi'(u) = 2u^2 \implies \text{Entropy function: } \psi(u) = \frac{2}{3}u^3 \).

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Weak solutions and entropy functions

The conservation laws

\[
\begin{align*}
  u_t + \left( \frac{1}{2}u^2 \right)_x &= 0 \quad \text{and} \\
  (u^2)_t + \left( \frac{2}{3}u^3 \right)_x &= 0
\end{align*}
\]

both have the same quasilinear form

\[
u_t + uu_x = 0
\]

but have different weak solutions, different shock speeds!

Entropy function: \( \eta(u) = u^2 \).

A correct Burgers’ shock at speed \( s = \frac{1}{2}(u_l + u_r) \) will have total mass of \( \eta(u) \) decreasing.

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Entropy functions

\[
\int_{x_1}^{x_2} \eta(q(x,t_2)) \, dx \leq \int_{x_1}^{x_2} \eta(q(x,t_1)) \, dx + \int_{t_1}^{t_2} \psi(q(x_1,t)) \, dt - \int_{t_1}^{t_2} \psi(q(x_2,t)) \, dt
\]

comes from considering the vanishing viscosity solution:

\[
q_t^\varepsilon + f(q^\varepsilon)_x = \varepsilon q_{xx}^\varepsilon
\]

Multiply by \( \eta'(q^\varepsilon) \) to obtain:

\[
\eta(q^\varepsilon)_t + \psi(q^\varepsilon)_x = \varepsilon \eta'(q^\varepsilon)q_{xx}^\varepsilon.
\]

Manipulate further to get

\[
\eta(q^\varepsilon)_t + \psi(q^\varepsilon)_x = \varepsilon(\eta'(q^\varepsilon)q_{xx}^\varepsilon) - \varepsilon\eta''(q^\varepsilon)(q_{xx}^\varepsilon)^2.
\]
Entropy functions

Smooth solution to viscous equation satisfies
\[ \eta(q\epsilon)_t + \psi(q\epsilon)_x = \epsilon (\eta'(q\epsilon) q_x) - \epsilon \eta''(q\epsilon) (q_x)^2. \]

Integrating over rectangle \([x_1, x_2] \times [t_1, t_2]\) gives
\[ \int_{x_1}^{x_2} \eta(q\epsilon(x, t_2)) \, dx = \int_{x_1}^{x_2} \eta(q\epsilon(x, t_1)) \, dx \]
\[ - (\int_{t_1}^{t_2} \psi(q\epsilon(x_2, t)) \, dt - \int_{t_1}^{t_2} \psi(q\epsilon(x_1, t)) \, dt) \]
\[ + \epsilon \int_{t_1}^{t_2} \left[ \eta''(q(x_2, t)) q_x(x_2, t) - \eta''(q(x_1, t)) q_x(x_1, t) \right] \, dt \]
\[ - \epsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \eta''(q\epsilon) (q_x)^2 \, dx \, dt. \]

Let \(\epsilon \to 0\) to get result:
Term on third line goes to 0,
Term of fourth line is always \(\leq 0\).

Weak form of entropy condition:
\[ \int_0^\infty \int_{-\infty}^\infty [\phi_t \eta(q) + \phi_x \psi(q)] \, dx \, dt + \int_{-\infty}^\infty \phi(x, 0) \eta(q(x, 0)) \, dx \geq 0 \]
for all \(\phi \in C^1_0(\mathbb{R} \times \mathbb{R})\) with \(\phi(x, t) \geq 0\) for all \(x, t\).

Informally we may write
\[ \eta(q)_t + \psi(q)_x \leq 0. \]

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with
\[ q_t + f(q)_x = 0, \]
\[ F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with} \quad \mathcal{F}(\bar{q}, \tilde{q}) = f(\bar{q}) \]
and Lipschitz continuity of \(\mathcal{F}\).

If a sequence of discrete approximations converge to a function \(q(x, t)\) as the grid is refined, then this function is a weak solution of the conservation law.

Note:
Does not guarantee a sequence converges (need stability).
Two sequences might converge to different weak solutions.
Also need to satisfy an entropy condition.
Sketch of proof of Lax-Wendroff Theorem

Multiply the conservative numerical method

\[ Q_{n+1}^i = Q_n^i - \frac{\Delta t}{\Delta x} (F_{n+1/2}^i - F_{n-1/2}^i) \]

by \( \Phi_i^n \) to obtain

\[ \Phi_i^n Q_{n+1}^i = \Phi_i^n Q_n^i - \frac{\Delta t}{\Delta x} \Phi_i^n (F_{n+1/2}^i - F_{n-1/2}^i). \]

This is true for all values of \( i \) and \( n \) on each grid. Now sum over all \( i \) and \( n \geq 0 \) to obtain

\[ \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \Phi_i^n (Q_{n+1}^i - Q_n^i) = -\Delta t \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \Phi_i^n (F_{n+1/2}^i - F_{n-1/2}^i). \]

Use summation by parts to transfer differences to \( \Phi \) terms.

Sketch of proof of Lax-Wendroff Theorem

Obtain analog of weak form of conservation law:

\[ \Delta x \Delta t \left[ \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} \left( \frac{\Phi_i^n - \Phi_{i-1}^{n-1}}{\Delta t} \right) Q_i^n \right. \]

\[ + \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \left( \frac{\Phi_i^{n+1} - \Phi_i^n}{\Delta x} \right) F_{i-1/2}^n \]

\[ = -\Delta x \sum_{i=-\infty}^{\infty} \Phi_i^n Q_i^n. \]

Consider on a sequence of grids with \( \Delta x, \Delta t \to 0 \).

Show that any limiting function must satisfy weak form of conservation law.

Analog of Lax-Wendroff proof for entropy

Show that the numerical flux function \( F \) leads to a numerical entropy flux \( \Psi \)

such that the following discrete entropy inequality holds:

\[ \eta(Q_{i+1}^{n+1}) \leq \eta(Q_i^n) - \frac{\Delta t}{\Delta x} \left[ \Psi_{i+1/2}^n - \Psi_{i-1/2}^n \right]. \]

Then multiply by test function \( \Phi_i^n \), sum and use summation by parts to get discrete form of integral form of entropy condition.

\( \implies \) If numerical approximations converge to some function, then the limiting function satisfies the entropy condition.
Entropy consistency of Godunov’s method

For Godunov’s method, \( F(Q_{i-1}, Q_i) = f(Q^{\psi}_{i-1/2}) \)
where \( Q^{\psi}_{i-1/2} \) is the constant value along \( x_{i-1/2} \) in the Riemann solution.

Let \( \Psi^{n}_{i-1/2} = \psi(Q^{\psi}_{i-1/2}) \)

Discrete entropy inequality follows from Jensen’s inequality:

The value of \( \eta \) evaluated at the average value of \( \tilde{q}^{n} \) is less than or equal to the average value of \( \eta(\tilde{q}^{n}) \), i.e.,

\[
\eta \left( \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{q}^{n}(x, t_{n+1}) \, dx \right) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \eta(\tilde{q}^{n}(x, t_{n+1})) \, dx.
\]