Outline

Today:
- Finite volume methods
- Conservation form
- Godunov's method
- Upwind method for advection, linear system
- CFL condition

Next:
- High resolution methods

Reading: Chapters 5 and 6

Finite differences vs. finite volumes

Finite difference Methods
- Pointwise values $Q^n_i \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods
- Approximate cell averages: $Q^n_i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx$

Integral form of conservation law,
\[
\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t) \, dx = f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))
\]
leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.
Finite volume method

Based on cell averages:

\[ Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx \]

Update cell average by flux into and out of cell:

**Ex:** Upwind methods for advection equation \( q_t + u q_x = 0 \):

\[ Q_i^{n+1} = Q_i^n - \Delta t \frac{\Delta x}{\Delta x} (Q_i^n - Q_{i-1}^n) \]

Stencil:

\[
\begin{array}{cccc}
Q_i^n & Q_i^{n+1} & Q_{i+1}^n & Q_{i+1}^{n+1} \\
Q_{i-1}^n & & & \\
\end{array}
\]

Nonlinear scalar conservation laws

Burgers’ equation: \( u_t + \frac{1}{2} u u_x = 0 \).

Quasilinear form: \( u_t + u u_x = 0 \).

These are equivalent for smooth solutions, not for shocks!

Upwind methods for \( u > 0 \):

Conservative:

\[ U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left( \frac{1}{2} (U_i^n)^2 - (U_{i-1}^n)^2 \right) \]

Quasilinear:

\[ U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} U_i^n (U_i^n - U_{i-1}^n) \]

Ok for smooth solutions, not for shocks!

Importance of conservation form

Solution to Burgers’ equation using conservative upwind:

Solution to Burgers’ equation using quasilinear upwind:
Conservation form

The method

\[ Q_{n+1}^i = Q_n^i - \Delta t \frac{\Delta x}{\Delta x} (F_{n+1/2}^i - F_{n-1/2}^i) \]

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

\[ \Delta x \sum_i Q_{n+1}^i = \Delta x \sum_i Q_n^i - \Delta t \Delta x (F_+\infty - F_-\infty). \]

Note: an isolated shock must travel at the right speed!

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with

\[ q_t + f(q)_x = 0, \]

\[ F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with} \quad \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q}) \]

and Lipschitz continuity of \( \mathcal{F} \).

If a sequence of discrete approximations converge to a function \( q(x, t) \) as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges
Two sequences might converge to different weak solutions.
Also need stability and entropy condition.

Finite volume method

Based on cell averages:

\[ Q_n^i \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n) \, dx \]

Update cell average by flux into and out of cell:

Ex: Upwind methods for advection equation \( q_t + u q_x = 0 \):

\[ Q_{n+1}^i = Q_n^i - \frac{\Delta t}{\Delta x} (u Q_{n-1}^i - u Q_n^i) \]

\[ = Q_n^i - \frac{\Delta t}{\Delta x} (Q_n^i - Q_{n-1}^i) \]

Stencil:

\[
\begin{array}{c|c|c}
Q_{n+1}^i & Q_n^i & Q_{n-1}^i \\
\hline
q_i^- & \cdot & q_i^+ \\
\end{array}
\]

(x-t plane)
Godunov’s Method for $q_t + f(q)_x = 0$

1. Solve Riemann problems at all interfaces, yielding waves $W^{p}_{i-1/2}$ and speeds $s^{p}_{i-1/2}$, for $p = 1, 2, \ldots, m$.

Riemann problem: Original equation with piecewise constant data.

Godunov’s method $Q^n_i$ defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q^n_i \quad \text{for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Rightarrow$ Riemann problems.

$$\tilde{q}^n(x_{i-1/2}, t) \equiv q^\gamma(Q^n_{i-1}, Q^n_i) \quad \text{for } t > t_n.$$
Wave-propagation viewpoint

For linear system \( q_t + A q_x = 0 \), the Riemann solution consists of waves \( W^p \) propagating at constant speed \( \lambda^p \).

\[
Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} W^p \equiv \sum_{p=1}^{m} W^p_{i-1/2}.
\]

\[
Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ \lambda^2 W^2_{i-1/2} + \lambda^3 W^3_{i-1/2} + \lambda^1 W^1_{i+1/2} \right].
\]

First-order REA Algorithm

1. **Reconstruct** a piecewise constant function \( \tilde{q}^n(x, t_n) \) defined for all \( x \), from the cell averages \( Q_i^n \).

\[
\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all} \quad x \in C_i.
\]

2. **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain \( \tilde{q}^n(x, t_{n+1}) \) a time \( \Delta t \) later.

3. **Average** this function over each grid cell to obtain new cell averages

\[
Q_i^{n+1} = \frac{1}{\Delta x} \int_{C_i} \tilde{q}^n(x, t_{n+1})\,dx.
\]

Godunov’s method for advection

\( Q_i^n \) defines a piecewise constant function

\[
\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for} \quad x_{i-1/2} < x < x_{i+1/2}
\]

Discontinuities at cell interfaces \( \implies \) Riemann problems.

\[
\begin{align*}
    u > 0 & \quad & u < 0
\end{align*}
\]
First-order REA Algorithm

Cell averages and piecewise constant reconstruction:

After evolution:

Cell update

The cell average is modified by

$$u \Delta t \cdot (Q^n_{i-1} - Q^n_i)$$

So we obtain the upwind method

$$Q^{n+1}_i = Q^n_i - u \Delta t \frac{\Delta x}{\Delta x} (Q^n_i - Q^n_{i-1}).$$

Godunov (upwind) on acoustics

Data at time $t_n$:

$$\tilde{q}^n(x, t_n) = Q^n_i \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Solving Riemann problems for small $\Delta t$ gives solution:

$$\tilde{q}^n(x, t_{n+1}) = \begin{cases} 
Q^n_{i-1/2} & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\
Q^n_i & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\
Q^n_{i+1/2} & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t,
\end{cases}$$

So computing cell average gives:

$$Q^{n+1}_i = \frac{1}{\Delta x} \left[ c\Delta t Q^n_{i-1/2} + (\Delta x - 2c\Delta t) Q^n_i + c\Delta t Q^n_{i+1/2} \right].$$
Godunov (upwind) on acoustics

\[ Q^{n+1}_i = \frac{1}{\Delta x} \left[ c\Delta t Q^*_i - 1/2 + (\Delta x - 2c\Delta t)Q^n_i + c\Delta t Q^*_i + 1/2 \right]. \]

Solve Riemann problems:

\[ Q^n_i - Q^n_{i-1} = \Delta Q^n_{i-1/2} = W^1_{i-1/2} + W^2_{i-1/2} = a^1_{i-1/2}r^1 + a^2_{i-1/2}r^2, \]
\[ Q^n_{i+1} - Q^n_i = \Delta Q^n_{i+1/2} = W^1_{i+1/2} + W^2_{i+1/2} = a^1_{i+1/2}r^1 + a^2_{i+1/2}r^2, \]

The intermediate states are:

\[ Q^*_i - 1/2 = Q^n_i - W^2_{i-1/2}, \quad Q^*_i + 1/2 = Q^n_i + W^1_{i+1/2}. \]

So,

\[ Q^{n+1}_i = \frac{1}{\Delta x} \left[ c\Delta t Q^*_i + 1/2 + (\Delta x - 2c\Delta t)Q^n_i + c\Delta t Q^*_i + 1/2 \right] \]
\[ = Q^n_i + \frac{c\Delta t}{\Delta x} W^2_{i-1/2} + \frac{c\Delta t}{\Delta x} W^1_{i+1/2}. \]

General form for linear system with \( m \) equations:

\[ Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x} \left[ \sum_{\rho > 0} \lambda^\rho W^\rho_{i-1/2} + \sum_{\rho < 0} \lambda^\rho W^\rho_{i+1/2} \right] \]
\[ = Q^n_i - \frac{\Delta t}{\Delta x} \left[ \sum_{m=1}^p (\lambda^m)^+ W^m_{i-1/2} + \sum_{m=1}^p (\lambda^m)^- W^m_{i+1/2} \right] \]

Godunov (upwind) on acoustics

\[ Q^{n+1}_i = \frac{1}{\Delta x} \left[ c\Delta t Q^*_i - 1/2 + (\Delta x - 2c\Delta t)Q^n_i + c\Delta t Q^*_i + 1/2 \right] \]
\[ = \frac{1}{\Delta x} \left[ c\Delta t (Q^n_i - W^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q^n_i + c\Delta t (Q^n_i + W^1_{i+1/2}) \right] \]
\[ = Q^n_i - \frac{c\Delta t}{\Delta x} W^2_{i-1/2} + \frac{c\Delta t}{\Delta x} W^1_{i+1/2} \]
\[ = Q^n_i - \frac{\Delta t}{\Delta x} (cW^2_{i-1/2} + (-c)W^1_{i+1/2}). \]

Notes:

Godunov (upwind) on acoustics

Solve Riemann problems:

\[ Q^n_i - Q^n_{i-1} = \Delta Q^n_{i-1/2} = W^1_{i-1/2} + W^2_{i-1/2} = a^1_{i-1/2}r^1 + a^2_{i-1/2}r^2, \]
\[ Q^n_{i+1} - Q^n_i = \Delta Q^n_{i+1/2} = W^1_{i+1/2} + W^2_{i+1/2} = a^1_{i+1/2}r^1 + a^2_{i+1/2}r^2, \]

The waves are determined by solving for \( \alpha \) from \( R\alpha = \Delta Q \):

\[ A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}, \quad R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \quad R^{-1} = \frac{1}{2Z} \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}. \]

So

\[ \Delta Q = \begin{bmatrix} \Delta p \\ \Delta u \end{bmatrix} = \alpha^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z \\ 1 \end{bmatrix} \]

with

\[ \alpha^1 = \frac{1}{2Z} (-\Delta p + Z\Delta u), \quad \alpha^2 = \frac{1}{2Z} (\Delta p + Z\Delta u). \]
Matrix splitting

Recall $A = R\Lambda R^{-1}$ with $\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$.

Let $\Lambda^+ = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$, \quad $\Lambda^- = \begin{bmatrix} -c & 0 \\ 0 & 0 \end{bmatrix}$.

and $A^+ = RA^+R^{-1}$, \quad $A^- = RA^-R^{-1}$.

Then $A^+ + A^- = R(\Lambda^+ + \Lambda^-)R^{-1} = A\Lambda R^{-1} = A$.

$A^+ \Delta Q = RA^+R^{-1}\Delta Q = RA^+Q = \sum_{p=1}^m (\lambda^p)^+\alpha^p r^p$

and similarly, $A^- \Delta Q = \sum_{p=1}^m (\lambda^p)^-\alpha^p r^p$.

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$Q^i_{n+1} = Q^i_n + \frac{\Delta t}{\Delta x} \left[ \sum_{p=1}^m (\lambda^p)^+\alpha_{i-1/2}^p r^p + \sum_{p=1}^m (\lambda^p)^-\alpha_{i+1/2}^p r^p \right]$$

Natural generalization of upwind to a system.

If all eigenvalues are positive, then $A^+ = A$ and $A^- = 0$,

If all eigenvalues are negative, then $A^+ = 0$ and $A^- = A$.

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $W^p$ propagating at constant speed $\lambda^p$.

$$Q_i - Q_{i-1} = \sum_{p=1}^m \alpha_{i-1/2}^p r^p \equiv \sum_{p=1}^m W^p_{i-1/2}.$$  

$$Q^n_{i+1} = Q^n_i - \frac{\Delta t}{\Delta x} \left[ \lambda^2 W_{i-1/2}^2 + \lambda^1 W_{i-1/2}^1 + \lambda^1 W_{i+1/2}^1 \right].$$
The CFL Condition

**Domain of dependence:** The solution \( q(X, T) \) depends on the data \( q(x, 0) \) over some set of \( x \) values, \( x \in D(X, T) \).

**Advection:** \( q(X, T) = q(X - uT, 0) \) and so \( D(X, T) = \{ X - uT \} \).

**The CFL Condition:** A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as \( \Delta t \) and \( \Delta x \) go to zero.

Note: Necessary but not sufficient for stability!

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**Numerical domain of dependence**

With a 3-point explicit method:

![Numerical domain of dependence diagram](image)

On a finer grid with \( \Delta t/\Delta x \) fixed:

![Numerical domain of dependence diagram](image)

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The CFL Condition

For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

\[ q(x, t) = q(x - ut, 0) \]

For a 3-point method, CFL condition requires \( \left| \frac{u \Delta x}{\Delta t} \right| \leq 1 \).

**If this is violated:**

True solution is determined by data at a point \( x - ut \) that is ignored by the numerical method, even as the grid is refined.

![CFL Condition diagram](image)
Stencil CFL Condition

\begin{align*}
0 \leq \frac{u \Delta t}{\Delta x} \leq 1 \\
-1 \leq \frac{u \Delta t}{\Delta x} \leq 0 \\
-1 \leq \frac{u \Delta t}{\Delta x} \leq 1 \\
0 \leq \frac{u \Delta t}{\Delta x} \leq 2 \\
-\infty < \frac{u \Delta t}{\Delta x} < \infty
\end{align*}

Notes:

Linear hyperbolic systems

Linear system of \( m \) equations: \( q(x, t) \in \mathbb{R}^m \) for each \((x, t)\) and
\[
\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0,
\quad -\infty < x, \infty, \quad t \geq 0.
\]

\( A \) is \( m \times m \) with eigenvalues \( \lambda^p \) and eigenvectors \( r^p \),
for \( p = 1, 2, \ldots, m \):
\[
A r^p = \lambda^p r^p.
\]

Combining these for \( p = 1, 2, \ldots, m \) gives
\[
AR = RA
\]

where
\[
R = \begin{bmatrix} r^1 & r^2 & \cdots & r^m \end{bmatrix}, \quad \Lambda = \text{diag}(\lambda^1, \lambda^2, \ldots, \lambda^m).
\]

The system is hyperbolic if the eigenvalues are real and
\( R \) is invertible. Then \( A \) can be diagonalized:
\[
R^{-1}AR = \Lambda
\]

Notes: