Today:
  • Approximate Riemann solvers
  • Multidimensional

Friday:
  • AMR

Projects: Make an appointment this week, and see
  http://www.clawpack.org/links/burgersadv
Shallow water equations

\[ h(x, t) = \text{depth} \]
\[ u(x, t) = \text{velocity (depth averaged, varies only with } x) \]

Conservation of mass and momentum \( hu \) gives system of two equations.

mass flux = \( hu \),
momentum flux = \( (hu)u + p \) where \( p = \text{hydrostatic pressure} \)

\[
\begin{align*}
    h_t + (hu)_x &= 0 \\
    (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x &= 0
\end{align*}
\]

Jacobian matrix:

\[
f'(q) = \begin{bmatrix}
0 & 1 \\
gh - u^2 & 2u
\end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.
\]
Given \( h_l, u_l, h_r, u_r \), define

\[
\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l u_l} + \sqrt{h_r u_r}}{\sqrt{h_l} + \sqrt{h_r}}
\]

Then

\[
\hat{A} = \text{Jacobian matrix evaluated at this average state}
\]

satisfies

\[
A(q_r - q_l) = f(q_r) - f(q_l).
\]

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.
Roe solver for Shallow Water

Given $h_l$, $u_l$, $h_r$, $u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l} u_l + \sqrt{h_r} u_r}{\sqrt{h_l} + \sqrt{h_r}}$$

Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g \bar{h}}.$$

Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \quad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$  

Examples in Clawpack 4.3 to be converted soon!
Potential failure of linearized solvers

Consider shallow water with $h_\ell = h_r$ and $u_r = -u_\ell \gg 1$.

Outflow away from interface $\implies$ small intermediate $h_m$.

With $u_r = 0.8$

Roe $h_m > 0$

With $u_r = 1.8$

Roe $h_m < 0$
Harten – Lax – van Leer (1983): Use only 2 waves with

- \( s^1 \) = minimum characteristic speed
- \( s^2 \) = maximum characteristic speed

\[ W^1 = Q^* - Q_\ell, \quad W^2 = Q_r - Q^* \]

Conservation implies unique value for middle state \( Q^* \):

\[ s^1 W^1 + s^2 W^2 = f(Q_r) - f(Q_\ell) \]

\[ \implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}. \]
HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with

\[ s^1 = \text{minimum characteristic speed} \]
\[ s^2 = \text{maximum characteristic speed} \]

\[ \mathcal{W}^1 = Q^* - Q_\ell, \quad \mathcal{W}^2 = Q_r - Q^* \]

Conservation implies unique value for middle state \( Q^* \):

\[ s^1 \mathcal{W}^1 + s^2 \mathcal{W}^2 = f(Q_r) - f(Q_\ell) \]
\[ \implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}. \]

Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of \( f'(Q_\ell) \) and \( f'(Q_r) \).
Einfieldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

\[ s_{i-1/2}^1 = \min_p (\min(\lambda_i^p, \hat{\lambda}_{i-1/2}^p)), \]
\[ s_{i-1/2}^2 = \max_p (\max(\lambda_{i+1}^p, \hat{\lambda}_{i-1/2}^p)). \]

where

\( \lambda_i^p \) is the \( p \)th eigenvalue of the Jacobian \( f'(Q_i) \),
\( \hat{\lambda}_{i-1/2}^p \) is the \( p \)th eigenvalue using Roe average \( f'(\hat{Q}_{i-1/2}) \).
Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}_i^{i-1/2}$:
  \[
  Q_i - Q_{i-1} = \sum_p \mathcal{W}_i^{i-1/2}
  \]
  and speeds $s_i^{i-1/2}$. (Usually approximate solver used.)

- These waves update neighboring cell averages depending on sign of $s^p$ (Godunov's method) via fluctuations.

- Waves also give characteristic decomposition of slopes:
  \[
  q_x(x_{i-1/2}, t) \approx \frac{Q_i - Q_{i-1}}{\Delta x} = \frac{1}{\Delta x} \sum_p \mathcal{W}_i^{i-1/2}
  \]

- Apply limiter to each wave to obtain $\tilde{\mathcal{W}}_i^{i-1/2}$.

- Use limited waves in second-order correction terms.
High-resolution wave-propagation algorithm

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (A^- \Delta Q_{i+1/2} + A^+ \Delta Q_{i-1/2}) - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}), \]

where

\[ \tilde{F}_{i-1/2} = \frac{1}{2} \sum_{p=1}^{m} |s_{i-1/2}^p| \left(1 - \frac{\Delta t}{\Delta x} |s_{i-1/2}^p| \right) \tilde{W}_{i-1/2}^p. \]

\( \tilde{W}_{i-1/2}^p \) represents a limited version of the wave \( W_{i-1/2}^p \), obtained by comparing this jump with the jump \( W_{I-1/2}^p \) in the same family at the neighboring Riemann problem in the upwind direction,

\[ I = \begin{cases} 
  i - 1 & \text{if } s_{i-1/2}^p > 0 \\
  i + 1 & \text{if } s_{i-1/2}^p < 0.
\end{cases} \]
Wave limiters

Let $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Upwind: $Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x} \mathcal{W}_{i-1/2}$.

Lax-Wendroff:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x} \mathcal{W}_{i-1/2} - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2} \left( 1 - \left| \frac{u\Delta t}{\Delta x} \right| \right) |u| \mathcal{W}_{i-1/2}$$

High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2} \left( 1 - \left| \frac{u\Delta t}{\Delta x} \right| \right) |u| \tilde{\mathcal{W}}_{i-1/2}$$

where $\tilde{\mathcal{W}}_{i-1/2} = \Phi_{i-1/2} \mathcal{W}_{i-1/2}$.
Extension to linear systems

**Approach 1:** Diagonalize the system to

\[ v_t + \Lambda v_x = 0 \]

Apply scalar algorithm to each component.

**Approach 2:**

Solve the linear Riemann problem to decompose \( Q_i^n - Q_{i-1}^n \) into waves.

Apply a wave limiter to each wave.

These are equivalent.

Important to apply limiters to waves or characteristic components, rather than to original variables.
Wave limiters for system

\( Q_i - Q_{i-1} \) is split into waves \( \mathcal{W}^{p}_{i-1/2} = \alpha^p_{i-1/2} r^p_{i-1/2} \in \mathbb{R}^m \).

Replace by \( \tilde{\mathcal{W}}^{p}_{i-1/2} = \Phi(\theta^p_{i-1/2}) \mathcal{W}^{p}_{i-1/2} \) where

\[
\theta^p_{i-1/2} = \frac{\mathcal{W}^{p}_{i-1/2} \cdot \mathcal{W}^{p}_{I-1/2}}{\mathcal{W}^{p}_{i-1/2} \cdot \mathcal{W}^{p}_{i-1/2}} = \frac{\alpha^p_{I-1/2}}{\alpha^p_{i-1/2}} \quad \text{if} \quad r^p_{i-1/2} = r^p_{I-1/2}
\]

where

\[
I = \begin{cases} 
  i - 1 & \text{if} \quad s^p_{i-1/2} > 0 \\
  i + 1 & \text{if} \quad s^p_{i-1/2} < 0.
\end{cases}
\]

In the scalar case this reduces to

\[
\theta^1_{i-1/2} = \frac{\mathcal{W}^{1}_{I-1/2}}{\mathcal{W}^{1}_{i-1/2}} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}
\]
First order hyperbolic PDE in 2 space dimensions

Advection equation: \[ q_t + uq_x + vq_y = 0 \]

First-order system: \[ q_t + Aq_x + Bq_y = 0 \]

where \( q \in \mathbb{R}^m \) and \( A, B \in \mathbb{R}^{m \times m} \).

Hyperbolic if \( \cos(\theta) A + \sin(\theta) B \) is diagonalizable with real eigenvalues, for all angles \( \theta \).
Advection equation: \( q_t + u q_x + v q_y = 0 \)

First-order system: \( q_t + A q_x + B q_y = 0 \)

where \( q \in \mathbb{R}^m \) and \( A, B \in \mathbb{R}^{m \times m} \).

Hyperbolic if \( \cos(\theta) A + \sin(\theta) B \) is diagonalizable with real eigenvalues, for all angles \( \theta \).

This is required so that plane-wave data gives a 1d hyperbolic problem:

\[
q(x, y, 0) = \breve{q}(x \cos \theta + y \sin \theta)
\]

implies contours of \( q \) in \( x-y \) plane are orthogonal to \( \theta \)-direction.
Acoustics in 2 dimensions

\[ \rho_0 u_t + p_x = 0 \]
\[ \rho_0 v_t + p_y = 0 \]

\[ A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \]

Solving \( q_t + A q_x = 0 \) gives pressure waves in \((p, u)\). 
\( x \)-variations in \( u \) are stationary.
Acoustics in 2 dimensions

\[ p_t + K_0(u_x + v_y) = 0 \]
\[ \rho_0 u_t + p_x = 0 \]
\[ \rho_0 v_t + p_y = 0 \]

\[
A = \begin{bmatrix}
0 & K_0 & 0 \\
1/\rho_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\quad
R^x = \begin{bmatrix}
-Z_0 & 0 & Z_0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

Solving \( qt + Aq_x = 0 \) gives pressure waves in \((p, u)\).
\( x \)-variations in \( v \) are stationary.

\[
B = \begin{bmatrix}
0 & 0 & K_0 \\
0 & 0 & 0 \\
1/\rho_0 & 0 & 0
\end{bmatrix}
\quad
R^y = \begin{bmatrix}
-Z_0 & 0 & Z_0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

Solving \( qt + Bq_y = 0 \) gives pressure waves in \((p, v)\).
\( y \)-variations in \( u \) are stationary.
2d finite volume method

\[ Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x} [F_{i+1/2,j}^n - F_{i-1/2,j}^n] \]
\[ - \frac{\Delta t}{\Delta y} [G_{i,j+1/2}^n - G_{i,j-1/2}^n]. \]

Fluctuation form:

\[ Q_{ij}^{n+1} = Q_{ij} - \frac{\Delta t}{\Delta x} (A^+ \Delta Q_{i-1/2,j} + A^- \Delta Q_{i+1/2,j}) \]
\[ - \frac{\Delta t}{\Delta y} (B^+ \Delta Q_{i,j-1/2} + B^- \Delta Q_{i,j+1/2}) \]
\[ - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y} (\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}). \]

The \( \tilde{F} \) and \( \tilde{G} \) are correction fluxes to go beyond Godunov’s upwind method.

Incorporate approximations to second derivative terms in each direction (\( q_{xx} \) and \( q_{yy} \)) and mixed term \( q_{xy} \).
Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver \texttt{rpn2.f}

Solves 1d Riemann problem \( q_t + A q_x = 0 \)

Decomposes \( \Delta Q = Q_{ij} - Q_{i-1,j} \) into \( A^+ \Delta Q \) and \( A^- \Delta Q \).

For \( q_t + A q_x + B q_y = 0 \), split using eigenvalues, vectors:

\[
A = R \Lambda R^{-1} \implies A^- = R \Lambda^- R^{-1}, \quad A^+ = R \Lambda^+ R^{-1}
\]

Input parameter \( \text{i} \times \text{y} \) determines if it’s in \( x \) or \( y \) direction.

In latter case splitting is done using \( B \) instead of \( A \).

This is all that’s required for dimensional splitting.
Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver rpn2.f
Solves 1d Riemann problem \( q_t + A q_x = 0 \)
Decomposes \( \Delta Q = Q_{ij} - Q_{i-1,j} \) into \( A^+ \Delta Q \) and \( A^- \Delta Q \).
For \( q_t + A q_x + B q_y = 0 \), split using eigenvalues, vectors:

\[
A = R \Lambda R^{-1} \implies A^- = R \Lambda^- R^{-1}, A^+ = R \Lambda^+ R^{-1}
\]

Input parameter \( ixy \) determines if it’s in \( x \) or \( y \) direction.
In latter case splitting is done using \( B \) instead of \( A \).
This is all that’s required for dimensional splitting.

Transverse Riemann solver rpt2.f
Decomposes \( A^+ \Delta Q \) into \( B^- A^+ \Delta Q \) and \( B^+ A^+ \Delta Q \) by splitting this vector into eigenvectors of \( B \).
(Or splits vector into eigenvectors of \( A \) if \( ixy=2 \).)
Acoustics in heterogeneous media

\[ q_t + A(x, y)q_x + B(x, y)q_y = 0, \quad q = (p, u, v)^T, \]

where

\[
A = \begin{bmatrix}
0 & K(x, y) & 0 \\
1/\rho(x, y) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & K(x, y) \\
0 & 0 & 0 \\
1/\rho(x, y) & 0 & 0
\end{bmatrix}.
\]

Note: Not in conservation form!
Acoustics in heterogeneous media

\[ q_t + A(x, y)q_x + B(x, y)q_y = 0, \quad q = (p, u, v)^T, \]

where

\[
A = \begin{bmatrix}
0 & K(x, y) & 0 \\
1/\rho(x, y) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & K(x, y) \\
0 & 0 & 0 \\
1/\rho(x, y) & 0 & 0
\end{bmatrix}.
\]

Note: Not in conservation form!

Wave propagation still makes sense. In \(x\)-direction:

\[
\mathcal{W}^1 = \alpha^1 \begin{bmatrix}
-Z_{i-1,j} \\
1 \\
0
\end{bmatrix}, \quad \mathcal{W}^2 = \alpha^2 \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad \mathcal{W}^3 = \alpha^3 \begin{bmatrix}
Z_{ij} \\
1 \\
0
\end{bmatrix}.
\]

Wave speeds: \(s_{i-1/2,j}^1 = -c_{i-1,j}, \quad s_{i-1/2,j}^2 = 0, \quad s_{i-1/2,j}^3 = +c_{ij}.\)
Acoustics in heterogeneous media

\[ W^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \quad W^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad W^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}. \]

Decompose \( \Delta Q = (\Delta p, \Delta u, \Delta v)^T \):

\[
\begin{align*}
\alpha_{i-1/2,j}^1 &= (-\Delta Q^1 + Z \Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\
\alpha_{i-1/2,j}^2 &= \Delta Q^3, \\
\alpha_{i-1/2,j}^3 &= (\Delta Q^1 + Z_{i-1,j} \Delta Q^2)/(Z_{i-1,j} + Z_{ij}).
\end{align*}
\]
Acoustics in heterogeneous media

\[ W^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \quad W^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad W^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}. \]

Decompose \( \Delta Q = (\Delta p, \Delta u, \Delta v)^T \):

\[ \alpha_{i-1/2,j}^1 = \frac{-\Delta Q^1 + Z \Delta Q^2}{Z_{i-1,j} + Z_{ij}}, \]
\[ \alpha_{i-1/2,j}^2 = \Delta Q^3, \]
\[ \alpha_{i-1/2,j}^3 = \frac{\Delta Q^1 + Z_{i-1,j} \Delta Q^2}{Z_{i-1,j} + Z_{ij}}. \]

Fluctuations:  \( \text{(Note: } s^1 < 0, \ s^2 = 0, \ s^3 > 0) \)

\[ A^- \Delta Q_{i-1/2,j} = s_{i-1/2,j}^1 W_{i-1/2,j}^1, \]
\[ A^+ \Delta Q_{i-1/2,j} = s_{i-1/2,j}^3 W_{i-1/2,j}^3. \]
Acoustics in heterogeneous media

Transverse solver: Split right-going fluctuation

\[ A^+ \Delta Q_{i-1/2,j} = s^3_{i-1/2,j} \mathcal{W}^3_{i-1/2,j} \]

into up-going and down-going pieces:

\[ \begin{align*}
B^+ A^+ \Delta Q &= \\
B^- A^+ \Delta Q &= 
\end{align*} \]

Decompose \( A^+ \Delta Q_{i-1/2,j} \) into eigenvectors of \( B \). Down-going:

\[ A^+ \Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{ij} \\ 0 \\ 1 \end{bmatrix}, \]
Transverse solver for acoustics

Up-going part: \( B^+ A^+ \Delta Q_{i-1/2,j} = c_{i,j+1} \beta^3 r^3 \) from

\[ A^+ \Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix}, \]

\[ \beta^3 = \left( (A^+ \Delta Q_{i-1/2,j})^1 + (A^+ \Delta Q_{i-1/2,j})^3 Z_{i,j+1} \right) / (Z_{ij} + Z_{i,j+1}). \]
Transverse Riemann solver in Clawpack

\texttt{rpt2} takes vector \texttt{asdq} and returns \texttt{bmasdq} and \texttt{bpasdq} where

\[
\text{asdq} = \mathbb{A}^* \Delta Q \text{ represents either } \\
\mathbb{A}^- \Delta Q \text{ if } \text{imp} = 1, \text{ or } \\
\mathbb{A}^+ \Delta Q \text{ if } \text{imp} = 2.
\]

Returns \[
\mathbb{B}^- \mathbb{A}^* \Delta Q \text{ and } \mathbb{B}^+ \mathbb{A}^* \Delta Q.
\]
rpt2 takes vector asdq and returns bmasdq and bpasdq where

\[ \text{asdq} = A^* \Delta Q \]

represents either
\[ A^- \Delta Q \] if \( \text{imp} = 1 \), or
\[ A^+ \Delta Q \] if \( \text{imp} = 2 \).

Returns \( B^- A^* \Delta Q \) and \( B^+ A^* \Delta Q \).

Note: there is also a parameter \( ixy \):

\( ixy = 1 \) means normal solve was in \( x \)-direction,
\( ixy = 2 \) means normal solve was in \( y \)-direction.

In this case \( \text{asdq} \) represents \( B^- \Delta Q \) or \( B^+ \Delta Q \) and the routine must return \( A^- B^* \Delta Q \) and \( A^+ B^* \Delta Q \).
Shallow water equations in two dimensions

\[ \begin{align*}
  &h_t + (hu)_x + (hv)_y = 0 \\
  & (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x + (huv)_y = 0 \\
  & (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2}gh^2 \right)_y = 0
\end{align*} \]
Shallow water equations in two dimensions

\[ h_t + (hu)_x + (hv)_y = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x + (huv)_y = 0 \]
\[ (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2}gh^2 \right)_y = 0 \]

Jacobian matrices:

\[ f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}, \quad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix}. \]
Shallow water equations in two dimensions

\[ h_t + (hu)_x + (hv)_y = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x + (huv)_y = 0 \]
\[ (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2}gh^2 \right)_y = 0 \]

Jacobian matrices:

\[
 f'(q) = \begin{bmatrix}
 0 & 1 & 0 \\
 -u^2 + gh & 2u & 0 \\
 -uv & v & u \\
\end{bmatrix}, 
 g'(q) = \begin{bmatrix}
 0 & 0 & 1 \\
 -uv & v & u \\
 -v^2 + gh & 0 & 2v \\
\end{bmatrix}.
\]

Eigenvalue and eigenvectors of \( f'(q) \):

\[ \lambda^{x_1} = u - c, \quad \lambda^{x_2} = u, \quad \lambda^{x_3} = u + c, \]
\[ r^{x_1} = \begin{bmatrix} 1 \\ u - c \end{bmatrix}, \quad r^{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad r^{x_3} = \begin{bmatrix} 1 \\ u + c \end{bmatrix}. \]
Jump in shear velocity $v$ is advected with velocity $u_m$. (Linearly degenerate)

Note: Variations in $v$ ($y$-velocity) in the $x$-direction do not compress fluid.
Structure of dam-break Riemann solution in 2d

\[ h_l > h_r, \quad u_l = u_r = 0, \quad v_l < 0, \quad v_r > 0. \]

Jump in shear velocity \( v \) is advected with velocity \( u_m \). (Linearly degenerate)

Note: Variations in \( v \) (y-velocity) in the \( x \)-direction do not compress fluid.

(Elasticity: restoring force from shear deformation \( \Rightarrow \) shear waves.)
Shallow water equations in two dimensions

\[
\begin{align*}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + \left( hu^2 + \frac{1}{2} gh^2 \right)_x + (huv)_y &= 0 \\
(hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2} gh^2 \right)_y &= 0
\end{align*}
\]
Shallow water equations in two dimensions

\begin{align*}
    h_t + (hu)_x + (hv)_y &= 0 \\
    (hu)_t + \left( hu^2 + \frac{1}{2} gh^2 \right)_x + (huv)_y &= 0 \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2} gh^2 \right)_y &= 0
\end{align*}

Roe averages:

\[
    \bar{h} = \frac{1}{2}(h_l + h_r), \quad \hat{u} = \frac{\sqrt{h_l} u_l + \sqrt{h_r} u_r}{\sqrt{h_l} + \sqrt{h_r}}, \hat{v} = \text{similar}.
\]

Roe matrix in $x$-direction:

\[
    \hat{A} = \begin{bmatrix}
        0 & 1 & 0 \\
        -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\
        -\hat{u}\hat{v} & \hat{v} & \hat{u}
    \end{bmatrix} = f'(\hat{q})
\]
Shallow water equations in two dimensions

Roe matrix in $x$-direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix},$$

has eigenvalues and eigenvectors

$$\hat{\lambda}^{x1} = \hat{u} - \hat{c}, \quad \hat{\lambda}^{x2} = \hat{u}, \quad \hat{\lambda}^{x3} = \hat{u} + \hat{c}$$

$$\hat{\rho}^{x1} = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \\ \hat{v} \end{bmatrix}, \quad \hat{\rho}^{x2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{\rho}^{x3} = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \\ \hat{v} \end{bmatrix}$$
Roe matrix in $x$-direction:

$$
\hat{A} = \begin{bmatrix}
0 & 1 & 0 \\
-\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\
-\hat{u}\hat{v} & \hat{v} & \hat{u}
\end{bmatrix},
$$

has eigenvalues and eigenvectors

\begin{align*}
\hat{\lambda}^{x1} &= \hat{u} - \hat{c}, \\
\hat{\lambda}^{x2} &= \hat{u}, \\
\hat{\lambda}^{x3} &= \hat{u} + \hat{c} \\
\hat{\mathbf{r}}^{x1} &= \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \\ \hat{v} \end{bmatrix}, \\
\hat{\mathbf{r}}^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
\hat{\mathbf{r}}^{x3} &= \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \\ \hat{v} \end{bmatrix}
\end{align*}

**Transverse solver:** use $\hat{v} \pm \hat{c}$ for transverse wave speeds.

http://www.amath.washington.edu/~claw/applications/shallow/2d/rp/