Today:
• Finite volume methods for nonlinear systems
• Wave propagation algorithms
• Approximate Riemann solvers

Wednesday:
• More about finite volume methods

Friday:
• Projects, What else??

Reading: Chapter 15
Projects: Make an appointment this week, and see
http://www.clawpack.org/links/burgersadv

Godunov’s method on a nonlinear system

Solve Riemann problems and average solution after time $\Delta t$.

$s_{\text{max}}\Delta t/\Delta x < 1/2$

$1/2 < s_{\text{max}}\Delta t/\Delta x < 1$

We do not want to compute nonlinear interaction of waves!

But can compute averages from edge fluxes without doing so!

Or with wave-propagation algorithm...

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ \sum_{p=1}^{m} (\lambda_p)^+ W_{i-1/2}^p + \sum_{p=1}^{m} (\lambda_p)^- W_{i+1/2}^p \right]$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ A^- \Delta Q_{i-1/2} + A^+ \Delta Q_{i+1/2} \right].$$

where the fluctuations are defined by

$$A^- \Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda_p)^- W_{i-1/2}^p,$$  left-going

$$A^+ \Delta Q_{i+1/2} = \sum_{p=1}^{m} (\lambda_p)^+ W_{i+1/2}^p,$$  right-going
All shock solution to the nonlinear Riemann problem

For the wave-propagation algorithm we need jump discontinuities $W_{i-1/2}^p$.

All-shock Riemann solution: Ignore rarefaction waves and use intersections of Hugoniot loci to define Riemann solution. Correct solution in some cases.

Will replace rarefaction waves by entropy-violating shocks.

If rarefaction is not transonic this is generally not a bad approximation: cell averages are very similar.

Transonic rarefactions can be handled by modifying $A^\pm \Delta Q_{i-1/2}$, the flux-difference splitting used in 1st order terms. Still use shock waves for high-resolution corrections.

Upwind wave-propagation algorithm

First order Godunov method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2} \right]$$

where

$$A^- \Delta Q_{i-1/2} = \sum_{p=1}^m (s_{i-1/2})^p W_{i-1/2}^p,$$

$$A^+ \Delta Q_{i+1/2} = \sum_{p=1}^m (s_{i+1/2})^p W_{i+1/2}^p.$$

May need to modify these by an entropy fix.

Entropy fix

Various approaches possible.

1. Compute "exact" value $q^\vee(Q_{i-1}, Q_i)$ and set

$$A^- \Delta Q_{i-1/2} = f(q^\vee) - f(Q_{i-1}),$$

$$A^+ \Delta Q_{i+1/2} = f(Q_i) - f(q^\vee).$$

2. Split transonic wave $W_{i-1/2}^p$ between $A^- \Delta Q_{i-1/2}$ and $A^+ \Delta Q_{i+1/2}$. 

Notes:

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AMath 574, March 7, 2011 [FVMHP Chap. 15]

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AMath 574, March 7, 2011 [FVMHP Sec. 15.3]

Notes:

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AMath 574, March 7, 2011 [FVMHP Sec. 15.3.5]
Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive. Often the nonlinear problem \( q_t + f(q)_x = 0 \) is approximated by
\[
q_t + A_{i-1/2} q_x = 0, \quad q_e = Q_{i-1}, \quad q_r = Q_i
\]
for some choice of \( A_{i-1/2} \approx f'(q) \) based on data \( Q_{i-1}, Q_i \).

Solve linear system for \( \alpha_{i-1/2}: Q_i - Q_{i-1} = \sum_p \alpha_{i-1/2}^p r_{i-1/2}^p \).

Waves \( W_{i-1/2}^p = \alpha_{i-1/2}^p r_{i-1/2}^p \) propagate with speeds \( s_{i-1/2}^p \),
\( r_{i-1/2}^p \) are eigenvectors of \( A_{i-1/2} \),
\( s_{i-1/2}^p \) are eigenvalues of \( A_{i-1/2} \).

Local linearization:
Replace \( q_t + f(q)_x = 0 \) by
\[
q_t + \hat{A} q_x = 0,
\]
where \( \hat{A} = \hat{A}(q_e, q_r) \approx f'(q_{ave}) \).

Then decompose
\[
q_r - q_l = \alpha^1 r^1 + \cdots + \alpha^m r^m
\]
to obtain waves \( W^p = \alpha^p r^p \) with speeds \( s^p = \lambda^p \).

Roe conditions for consistency and conservation:
- \( \hat{A}(q_e, q_r) \to f'(q^*) \) as \( q_e, q_r \to q^* \),
- \( \hat{A} \) diagonalizable with real eigenvalues,
- For conservation in wave-propagation form,
  \[
  \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).
  \]
Roe Solver

Solve $q_t + \hat{A}q_x = 0$ where $\hat{A}$ satisfies

$$\hat{A}(q_r - q_l) = f(q_r) - f(q_l).$$

Then:

- Good approximation for weak waves (smooth flow)
- Single shock captured exactly:
  $$f(q_r) - f(q_l) = s(q_r - q_l) \implies q_r - q_l \text{ is an eigenvector of } \hat{A}$$
- Wave-propagation algorithm is conservative since
  $$\hat{A}^{-} \Delta Q_{i-1/2} + \hat{A}^{+} \Delta Q_{i-1/2} = \sum_{p: s_p < 0} s_p W^p_{i-1/2} = A \sum W^p_{i-1/2}.$$  

Roe average $\hat{A}$ can be determined analytically for many important nonlinear systems (e.g. Euler, shallow water).

Notes:

Approximate solution to single wave

Suppose $q_l$ lies on some Hugoniot locus of $q_r$ (and vice versa):

$$\hat{Q}_{i-1/2} = \frac{1}{2} (Q_{i-1} + Q_i) \quad \hat{Q}_{i+1/2} = \text{Roe average}$$

Straight lines are eigendirections of $f'(\hat{Q}_{i-1/2})$.

Notes:

Approximate Riemann Solvers

How to use?

One approach: determine $Q^* = \text{state along } x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p: s_p < 0} W^p, \quad F_{i-1/2} = f(Q^*),$$

$$\hat{A}^{-} \Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \hat{A}^{+} \Delta Q = f(Q_i) - F_{i-1/2}.$$  

Wave-propagation algorithm uses:

$$\hat{A}^{-} \Delta Q = \sum_{p: s_p < 0} s^p W^p, \quad \hat{A}^{+} \Delta Q = \sum_{p: s_p > 0} s^p W^p.$$  

Conservative only if $\hat{A}^{-} \Delta Q + \hat{A}^{+} \Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.
Approximate Riemann solvers

For a scalar problem, we can easily satisfy the Roe condition
\[ \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}) \]
by choosing
\[ \hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}. \]

Then \( r_{i-1/2}^1 = 1 \) and \( s_{i-1/2}^1 = \hat{A}_{i-1/2} \) (scalar!).

**Note:** This is the Rankine-Hugoniot shock speed.

\[ \implies \text{shock waves are correct, rarefactions replaced by entropy-violating shocks.} \]

Shallow water equations

\[ h(x, t) = \text{depth} \]
\[ u(x, t) = \text{velocity (depth averaged, varies only with } x) \]

Conservation of mass and momentum \( hu \) gives system of two equations.

mass flux = \( hu \),
momentum flux = \( (hu)u + p \) where \( p \) = hydrostatic pressure

\[ h_t + (hu)_x = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x = 0 \]

Jacobian matrix:
\[ f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}. \]

Roe solver for Shallow Water

Given \( h_l, u_l, h_r, u_r, \) define
\[ \bar{h} = \frac{h_l + h_r}{2}, \quad \bar{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}} \]

Then
\[ \hat{A} = \text{Jacobian matrix evaluated at this average state} \]
satisfies
\[ A(q_r - q_l) = f(q_r) - f(q_l). \]

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.
Roe solver for Shallow Water

Given \( h_l, u_l, h_r, u_r \), define
\[
\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \sqrt{\frac{h_l u_l + h_r u_r}{h_l + h_r}}
\]

Eigenvalues of \( \hat{A} = f'(\hat{q}) \) are:
\[
\hat{\lambda}_1 = \hat{u} - \hat{c}, \quad \hat{\lambda}_2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{gh}.
\]

Eigenvectors:
\[
\hat{r}_1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \quad \hat{r}_2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.
\]

Examples in Clawpack 4.3 to be converted soon!

Potential failure of linearized solvers

Consider shallow water with \( h_l = h_r \) and \( u_r = -u_l \gg 1 \).
Outflow away from interface \( \implies \) small intermediate \( h_m \).

With \( u_r = 0.8 \) \( \implies \) Roe \( h_m > 0 \)
With \( u_r = 1.8 \) \( \implies \) Roe \( h_m < 0 \)

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with
\( s^1 \) = minimum characteristic speed
\( s^2 \) = maximum characteristic speed
\[
W^1 = Q^* - Q_l, \quad W^2 = Q_r - Q^*
\]

Conservation implies unique value for middle state \( Q^* \):
\[
s^1 W^1 + s^2 W^2 = f(Q_r) - f(Q_l)
\]
\[
\implies Q^* = \frac{f(Q_r) - f(Q_l) - s^2 Q_r + s^1 Q_l}{s^1 - s^2}.
\]

Choice of speeds:
- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of \( f'(Q_l) \) and \( f'(Q_r) \).
Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

\[
\begin{align*}
    s_{i-1/2}^1 &= \min_p \left( \min\left( \lambda_p^i, \hat{\lambda}_{i-1/2}^p \right) \right), \\
    s_{i-1/2}^2 &= \max_p \left( \max\left( \lambda_{i+1}^p, \hat{\lambda}_{i-1/2}^p \right) \right),
\end{align*}
\]

where

- \( \lambda_p^i \) is the \( p \)-th eigenvalue of the Jacobian \( f'(Q_i) \),
- \( \hat{\lambda}_{i-1/2}^p \) is the \( p \)-th eigenvalue using Roe average \( f'((\hat{Q}_{i-1/2}) \).