Today:

- Finite volume methods for nonlinear systems
- Wave propagation algorithms
- Approximate Riemann solvers

Wednesday:

- More about finite volume methods

Friday:

- Projects, What else??

Reading: Chapter 15

Projects: Make an appointment this week, and see

http://www.clawpack.org/links/burgersadv
Godunov’s method on a nonlinear system

Solve Riemann problems and average solution after time $\Delta t$.

$s_{\text{max}} \Delta t/\Delta x < 1/2$

$1/2 < s_{\text{max}} \Delta t/\Delta x < 1$
Godunov’s method on a nonlinear system

Solve Riemann problems and average solution after time $\Delta t$.

We do not want to compute nonlinear interaction of waves!

\[ s_{\text{max}} \frac{\Delta t}{\Delta x} < \frac{1}{2} \]

\[ \frac{1}{2} < s_{\text{max}} \frac{\Delta t}{\Delta x} < 1 \]
Godunov’s method on a nonlinear system

Solve Riemann problems and average solution after time $\Delta t$.

\[ s_{\text{max}} \frac{\Delta t}{\Delta x} < \frac{1}{2} \]

\[ \frac{1}{2} < s_{\text{max}} \frac{\Delta t}{\Delta x} < 1 \]

We do not want to compute nonlinear interaction of waves!

But can compute averages from edge fluxes without doing so!

Or with wave-propagation algorithm...
Upwind wave-propagation algorithm

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ \sum_{p=1}^{m} (\lambda^{p})^{+} \mathcal{W}_{i-1/2}^{p} + \sum_{p=1}^{m} (\lambda^{p})^{-} \mathcal{W}_{i+1/2}^{p} \right] \]

or

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ \mathcal{A}^{+} \Delta Q_{i-1/2} + \mathcal{A}^{-} \Delta Q_{i+1/2} \right]. \]

where the fluctuations are defined by

\[ \mathcal{A}^{-} \Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^{p})^{-} \mathcal{W}_{i-1/2}^{p}, \quad \text{left-going} \]

\[ \mathcal{A}^{+} \Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^{p})^{+} \mathcal{W}_{i-1/2}^{p}, \quad \text{right-going} \]
All shock solution to the nonlinear Riemann problem

For the wave-propagation algorithm we need jump discontinuities \( W^{p}_{i-1/2} \).

**All-shock Riemann solution:** Ignore rarefaction waves and use intersections of Hugoniot loci to define Riemann solution.

Correct solution in some cases.
All shock solution to the nonlinear Riemann problem

For the wave-propagation algorithm we need jump discontinuities $\mathcal{W}_i^{p} - \frac{1}{2}$.

**All-shock Riemann solution:** Ignore rarefaction waves and use intersections of Hugoniot loci to define Riemann solution.

Correct solution in some cases.

Will replace rarefaction waves by **entropy-violating shocks**.

If rarefaction is **not transonic** this is generally not a bad approximation: cell averages are very similar.
All shock solution to the nonlinear Riemann problem

For the wave-propagation algorithm we need jump discontinuities $\mathcal{W}^p_{i-1/2}$.

**All-shock Riemann solution:** Ignore rarefaction waves and use intersections of Hugoniot loci to define Riemann solution.

Correct solution in some cases.

Will replace rarefaction waves by *entropy-violating shocks*.

If rarefaction is *not transonic* this is generally not a bad approximation: cell averages are very similar.

**Transonic rarefactions** can be handled by modifying $A^\pm \Delta Q_{i-1/2}$, the flux-difference splitting used in 1st order terms.

Still use shock waves for high-resolution corrections.
Upwind wave-propagation algorithm

First order Godunov method:

\[ Q_{i}^{n+1} = Q_{i}^{n} - \frac{\Delta t}{\Delta x} \left[ A^{+} \Delta Q_{i-1/2} + A^{-} \Delta Q_{i+1/2} \right] \]

where

\[ A^{-} \Delta Q_{i-1/2} = \sum_{p=1}^{m} (s_{i-1/2}^{p})^{-} \mathcal{W}_{i-1/2}^{p}, \]

\[ A^{+} \Delta Q_{i+1/2} = \sum_{p=1}^{m} (s_{i-1/2}^{p})^{+} \mathcal{W}_{i-1/2}^{p}, \]

May need to modify these by an entropy fix.
Various approaches possible.

1. Compute “exact” value \( q^\downarrow(Q_{i-1}, Q_i) \) and set

\[
\begin{align*}
A^- \Delta Q_{i-1/2} &= f(q^\downarrow) - f(Q_{i-1}), \\
A^+ \Delta Q_{i-1/2} &= f(Q_i) - f(q^\downarrow).
\end{align*}
\]

2. Split transonic wave \( \mathcal{W}^p_{i-1/2} \) between \( A^- \Delta Q_{i-1/2} \) and \( A^+ \Delta Q_{i-1/2} \).
Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive.

Often the nonlinear problem $q_t + f(q)_x = 0$ is approximated by

$$q_t + A_{i-1/2}q_x = 0, \quad q^\ell = Q_{i-1}, \quad q^r = Q_i$$

for some choice of $A_{i-1/2} \approx f'(q)$ based on data $Q_{i-1}, Q_i$. 

Waves $W_{p_i-1/2}$ propagate with speeds $s_{p_i-1/2}$, and $r_{p_i-1/2}$ are eigenvectors of $A_{i-1/2}$, $s_{p_i-1/2}$ are eigenvalues of $A_{i-1/2}$. 

R.J. LeVeque, University of Washington  

AMath 574, March 7, 2011  

[FVMHP Sec. 15.3.2 ]
Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive.

Often the nonlinear problem $q_t + f(q)_x = 0$ is approximated by

$$q_t + A_{i-1/2} q_x = 0, \quad q_\ell = Q_{i-1}, \quad q_r = Q_i$$

for some choice of $A_{i-1/2} \approx f'(q)$ based on data $Q_{i-1}, Q_i$.

Solve linear system for $\alpha_{i-1/2}$:

$$Q_i - Q_{i-1} = \sum_p \alpha_p^{i-1/2} r_p^{i-1/2}.$$  

Waves $\mathcal{W}_p^{i-1/2} = \alpha_p^{i-1/2} r_p^{i-1/2}$ propagate with speeds $s_p^{i-1/2}$,

- $r_p^{i-1/2}$ are eigenvectors of $A_{i-1/2}$,
- $s_p^{i-1/2}$ are eigenvalues of $A_{i-1/2}$. 

R.J. LeVeque, University of Washington  
AMath 574, March 7, 2011  
[FVMHP Sec. 15.3.2]
Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

**Local linearization:**

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave}).$

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \cdots \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p.$
Approximate Riemann solvers

\[ q_t + \hat{A}_{i-1/2} q_x = 0, \quad q_\ell = Q_{i-1}, \quad q_r = Q_i \]

Often \( \hat{A}_{i-1/2} = f'(Q_{i-1/2}) \) for some choice of \( Q_{i-1/2} \).

In general \( \hat{A}_{i-1/2} = \hat{A}(q_\ell, q_r) \).
Approximate Riemann solvers

\[ q_t + \hat{A}_{i-1/2} q_x = 0, \quad q_\ell = Q_{i-1}, \quad q_r = Q_i \]

Often \( \hat{A}_{i-1/2} = f'(Q_{i-1/2}) \) for some choice of \( Q_{i-1/2} \).

In general \( \hat{A}_{i-1/2} = \hat{A}(q_\ell, q_r) \).

Roe conditions for consistency and conservation:

1. \( \hat{A}(q_\ell, q_r) \to f'(q^*) \) as \( q_\ell, q_r \to q^* \),
2. \( \hat{A} \) diagonalizable with real eigenvalues,
3. For conservation in wave-propagation form,

\[
\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).
\]
Solve \( q_t + \hat{A} q_x = 0 \) where \( \hat{A} \) satisfies

\[
\hat{A}(q_r - q_l) = f(q_r) - f(q_l).
\]

Then:

- Good approximation for weak waves (smooth flow)
- Single shock captured exactly:
  \[
  f(q_r) - f(q_l) = s(q_r - q_l) \implies q_r - q_l \text{ is an eigenvector of } \hat{A}
  \]
- Wave-propagation algorithm is conservative since

\[
\mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} = \sum s_{i-1/2}^p \mathcal{W}_{i-1/2}^p = A \sum \mathcal{W}_{i-1/2}^p.
\]
Solve $q_t + \hat{A}q_x = 0$ where $\hat{A}$ satisfies

$$\hat{A}(q_r - q_l) = f(q_r) - f(q_l).$$

Then:

- Good approximation for weak waves (smooth flow)
- Single shock captured exactly:

$$f(q_r) - f(q_l) = s(q_r - q_l) \implies q_r - q_l \text{ is an eigenvector of } \hat{A}$$

- Wave-propagation algorithm is conservative since

$$A^- \Delta Q_{i-1/2} + A^+ \Delta Q_{i-1/2} = \sum s^p_{i-1/2} W^p_{i-1/2} = A \sum W^p_{i-1/2}.$$ 

Roe average $\hat{A}$ can be determined analytically for many important nonlinear systems (e.g. Euler, shallow water).
Approximate solution to single wave

Suppose $q_\ell$ lies on some Hugoniot locus of $q_r$ (and vice versa!):

\[ \hat{Q}_{i-1/2} = \frac{1}{2}(Q_{i-1} + Q_i) \]

\[ \hat{Q}_{i-1/2} = \text{Roe average} \]

Straight lines are eigendirections of $f'(\hat{Q}_{i-1/2})$. 
How to use?

One approach: determine $Q^* = \text{state along } x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p: s^p < 0} W^p, \quad F_{i-1/2} = f(Q^*),$$

$$A^- \Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad A^+ \Delta Q = f(Q_i) - F_{i-1/2}.$$
Approximate Riemann Solvers

How to use?

One approach: determine $Q^* = \text{state along } x/t = 0$, 

$$Q^* = Q_{i-1} + \sum_{p: s^p < 0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$A^− \Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad A^+ \Delta Q = f(Q_i) - F_{i-1/2}.$$

Wave-propagation algorithm uses:

$$A^− \Delta Q = \sum_{p: s^p < 0} s^p \mathcal{W}^p, \quad A^+ \Delta Q = \sum_{p: s^p > 0} s^p \mathcal{W}^p.$$

Conservative only if $A^− \Delta Q + A^+ \Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.
For a **scalar** problem, we can easily satisfy the Roe condition

\[ \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}). \]

by choosing

\[ \hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}. \]
For a **scalar** problem, we can easily satisfy the Roe condition

\[ \hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}). \]

by choosing

\[ \hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}. \]

Then \( r_{i-1/2}^1 = 1 \) and \( s_{i-1/2}^1 = \hat{A}_{i-1/2} \) (scalar!).

**Note:** This is the Rankine-Hugoniot shock speed.

\( \implies \) shock waves are correct,

rarefactions replaced by **entropy-violating shocks**.
Shallow water equations

\[ h(x, t) = \text{depth} \]
\[ u(x, t) = \text{velocity (depth averaged, varies only with } x) \]

Conservation of mass and momentum \(hu\) gives system of two equations.

mass flux = \(hu\),
momentum flux = \((hu)u + p\) where \(p = \text{hydrostatic pressure}\)

\[ h_t + (hu)_x = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x = 0 \]

Jacobian matrix:

\[ f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}. \]
Given $h_l$, $u_l$, $h_r$, $u_r$, define

$$
\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}
$$

Then

$$
\hat{A} = \text{Jacobian matrix evaluated at this average state}
$$

satisfies

$$
A(q_r - q_l) = f(q_r) - f(q_l).
$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.
Given $h_l$, $u_l$, $h_r$, $u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}$$

Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}_1 = \hat{u} - \hat{c}, \quad \hat{\lambda}_2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{gh}.$$ 

Eigenvectors:

$$\hat{r}_1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \quad \hat{r}_2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

Examples in Clawpack 4.3 to be converted soon!
Potential failure of linearized solvers

Consider shallow water with \( h_\ell = h_r \) and \( u_r = -u_\ell \gg 1 \).

Outflow away from interface \( \implies \) small intermediate \( h_m \).

With \( u_r = 0.8 \)

\[ \text{Roe } h_m > 0 \]

With \( u_r = 1.8 \)

\[ \text{Roe } h_m < 0 \]
Harten – Lax – van Leer (1983): Use only 2 waves with
\( s^1 \) = minimum characteristic speed
\( s^2 \) = maximum characteristic speed

\[ \mathcal{W}^1 = Q^* - Q_\ell, \quad \mathcal{W}^2 = Q_r - Q^* \]

Conservation implies unique value for middle state \( Q^* \):

\[ s^1 \mathcal{W}^1 + s^2 \mathcal{W}^2 = f(Q_r) - f(Q_\ell) \]

\[ \implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}. \]
Harten – Lax – van Leer (1983): Use only 2 waves with
\( s^1 = \) minimum characteristic speed
\( s^2 = \) maximum characteristic speed

\[ \mathcal{W}^1 = Q^* - Q_\ell, \quad \mathcal{W}^2 = Q_r - Q^* \]

Conservation implies unique value for middle state \( Q^* \):

\[ s^1 \mathcal{W}^1 + s^2 \mathcal{W}^2 = f(Q_r) - f(Q_\ell) \]

\[ \implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}. \]

Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of \( f'(Q_\ell) \) and \( f'(Q_r) \).
Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

\[
\begin{align*}
    s_{i-1/2}^1 &= \min_p (\min (\lambda_p^i, \hat{\lambda}_{i-1/2}^p)), \\
    s_{i-1/2}^2 &= \max_p (\max (\lambda_{i+1}^p, \hat{\lambda}_{i-1/2}^p)).
\end{align*}
\]

where

- \(\lambda_p^i\) is the \(p\)th eigenvalue of the Jacobian \(f'(Q_i)\),
- \(\hat{\lambda}_{i-1/2}^p\) is the \(p\)th eigenvalue using Roe average \(f'(\hat{Q}_{i-1/2})\).