Burgers’ + advection

Another example of a nonlinear system:

\[ q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u+1)v \end{bmatrix}. \]

This is simply Burgers’ equation

\[ u_t + \frac{1}{2}(u^2)_x = 0 \]

coupled to conservative advection

\[ v_t + ((u+1)v)_x = 0 \]

But... Advection velocity \( u + 1 \) comes from solution of Burgers’ equation.

Burgers’ + advection

Solving \( u_t + \frac{1}{2}(u^2)_x = 0 \) gives rarefaction wave (if \( u_l < u_r \)) or shock wave with speed \( s^1 = \frac{1}{2}(u_l + u_r) \) (if \( u_l > u_r \)).

Advection equation can be rewritten as

\[ v_t + (u + 1)v_x = -u_x v \]

and characteristic theory shows that

\[ \frac{d}{dt} v(X(t), t) = -u_x(X(t), t)v(X(t), t) \]

along the curve \( X'(t) = u(X(t), t) + 1. \)

In regions where \( u \) is constant:

- Characteristics are straight lines,
- \( u_x = 0 \implies v \) is constant.
Burgers’ + advection

\[
\frac{d}{dt} v(X(t), t) = -u_x(X(t), t) v(X(t), t)
\]

along the curve \( X'(t) = u(X(t), t) + 1 \).

If \( u \) has a shock, then source term in \( v \) has form of delta function.

If delta moves a different speed than advection velocity, this leads to a jump in \( v \) at the shock location.

**Resonant case:** If shock moves at same speed as advection velocity then delta function is stationary relative to advecting \( v \) and we expect solution to blow up!

Reconsider as nonlinear system:

\[
q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2} (u^2) \\ (u + 1)v \end{bmatrix}.
\]

Jacobian matrix:

\[
f'(q) = \begin{bmatrix} u & 0 \\ v & u + 1 \end{bmatrix}.
\]

Always hyperbolic since \( u \neq u + 1 \).

\[
\lambda_1 = u, \quad r_1 = \begin{bmatrix} 1 \\ -v \end{bmatrix}, \quad \nabla \lambda_1 \cdot r_1 \equiv 1, \quad \text{genuinely nonlinear}
\]

\[
\lambda_2 = u + 1, \quad r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla \lambda_2 \cdot r_2 \equiv 0, \quad \text{linearly degenerate}
\]

**Integral curves:**

\[
\tilde{u}'(\xi) = 0 \quad \Rightarrow \quad \tilde{u}(\xi) = u_*
\]

\[
\tilde{v}'(\xi) = v(\xi) \quad \Rightarrow \quad \tilde{v}(\xi) = v_* e^{\xi}
\]

Integral curves are vertical lines.

These lines are also contours of \( \lambda^2 \) (linearly degenerate!)

We’ll see later these are also the Hugoniot loci for 2-waves.
Burgers’ + advection: 1-waves

\[ \lambda^1 = u, \quad r^1 = \left[ \begin{array}{c} 1 \\ -v \end{array} \right], \quad \nabla \lambda^1 \cdot r^1 \equiv 1, \quad \text{genuinely nonlinear} \]

Integral curves:

\[ \tilde{\lambda}'(\xi) = 1 \quad \Rightarrow \quad \tilde{\lambda}(\xi) = u^* + \xi \quad \Rightarrow \quad \xi = \tilde{\lambda} - u^* \]

\[ \tilde{v}'(\xi) = -v(\xi) \quad \Rightarrow \quad \tilde{v}(\xi) = v_* e^{-\xi} \quad \Rightarrow \quad \tilde{v} = v_* e^{u_* \tilde{\lambda}}. \]

R.J. LeVeque, University of Washington AMath 574, February 28, 2011 [FVMHP Chap. 13]

Burgers’ + advection: Hugoniot loci

\[ q = \left[ \begin{array}{c} u \\ v \end{array} \right], \quad f(q) = \left[ \begin{array}{c} \frac{1}{2} u^2 \\ (u + 1)v \end{array} \right]. \]

States \( q \) and \( q_* \) must satisfy Rankine-Hugoniot jump condition:

\[ f(q) - f(q_*) = s(q - q_*). \]

First equation gives:

\[ \frac{1}{2}(u^2 - u_*^2) = s(u - u_*) \quad \Rightarrow \quad \frac{1}{2}(u + u_*)(u - u_*) = s(u - u_*). \]

One solution:

\[ u = u_* \quad \text{(and jump in } v \text{ arbitrary)} \quad \Rightarrow \quad \text{vertical lines} \]

These are Hugoniot loci for 2-waves.

2-waves are discontinuities in \( v \) alone, speed \( s = u_* + 1 \)

(determined from second equation of R-H conditions).

R.J. LeVeque, University of Washington AMath 574, February 28, 2011 [FVMHP Chap. 13]

Burgers’ + advection: Hugoniot loci

\[ \frac{1}{2}(u^2 - u_*^2) = s(u - u_*) \quad \Rightarrow \quad \frac{1}{2}(u + u_*)(u - u_*) = s(u - u_*). \]

Second solution:

\[ s = s^1 = \frac{1}{2}(u + u_*) \quad \Rightarrow \quad \text{shock waves in Burgers’ equation} \]

Relation between \( v \) and \( u \) across shock:

Second equation of R-H relation:

\[ (u + 1)v - (u_* + 1)v_* = s(v - v_*) = \frac{1}{2}(u + u_*)(v - v_*). \]

\[ \Rightarrow \quad v = \left( \frac{1 + \frac{1}{2}(u_* - u)}{1 - \frac{1}{2}(u_* - u)} \right) v_* \approx e^{u_* - u} v_* \]

The Hugoniot locus agrees to \( O(|u_* - u|^3) \) with integral curve.
But note that

\[ v = \left( 1 + \frac{1}{2} (u^* - u) \right) u^* \rightarrow \infty \quad \text{as} \quad u \rightarrow u^* - 2 \]