Today:
- Another example nonlinear system: Burgers’ + Advection
- Shallow water Riemann solution

Next Monday:
- Finite volume methods
- Approximate Riemann solvers

Reading: Chapter 15
Another example of a nonlinear system:

\[
q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u + 1)v \end{bmatrix}.
\]

This is simply Burgers’ equation

\[
\frac{\partial u}{\partial t} + \frac{1}{2}(u^2)_x = 0
\]

coupled to conservative advection

\[
\frac{\partial v}{\partial t} + ((u + 1)v)_x = 0
\]

But... Advection velocity \( u + 1 \) comes from solution of Burgers’ equation.
Solving $u_t + \frac{1}{2} (u^2)_x = 0$ gives rarefaction wave (if $u_l < u_r$) or shock wave with speed $s^1 = \frac{1}{2} (u_l + u_r)$ (if $u_l > u_r$).
Burgers’ + advection

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Advection equation can be rewritten as

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v_t + (u + 1) v_x = -u_x v
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and characteristic theory shows that

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\frac{d}{dt} v(X(t), t) = -u_x(X(t), t) v(X(t), t)
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along the curve \( X'(t) = u(X(t), t) + 1 \).
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In regions where \( u \) is constant:

Characteristics are straight lines,

\( u_x = 0 \implies v \) is constant.
Burgers’ + advection

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If \( u \) has a shock, then source term in \( v \) has form of delta function.

If delta moves a different speed than advection velocity, this leads to a jump in \( v \) at the shock location.

Resonant case: If shock moves at same speed as advection velocity then delta function is stationary relative to advecting \( v \) and we expect solution to blow up!
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Reconsider as nonlinear system:

\[ q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u + 1)v \end{bmatrix}. \]

Jacobian matrix:

\[ f'(q) = \begin{bmatrix} u & 0 \\ v & u + 1 \end{bmatrix}. \]

Always \textbf{hyperbolic} since \( u \neq u + 1 \).
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\[ \lambda^1 = u, \quad r^1 = \begin{bmatrix} 1 \\ -v \end{bmatrix}, \quad \nabla \lambda^1 \cdot r^1 \equiv 1, \quad \text{genuinely nonlinear} \]
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\lambda^2 = u + 1, \quad r^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla \lambda^2 \cdot r^2 \equiv 0, \quad \text{linearly degenerate}
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Burgers’ + advection: 2-waves

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Integral curves:

\[ \tilde{u}'(\xi) = 0 \quad \implies \quad \tilde{u}(\xi) = u_* \]

\[ \tilde{v}'(\xi) = v(\xi) \quad \implies \quad \tilde{v}(\xi) = v_* e^{\xi} \]

Integral curves are vertical lines.
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Integral curves are vertical lines.

These lines are also contours of \( \lambda^2 \) (linearly degenerate!)

We’ll see later these are also the Hugoniot loci for 2-waves.
Burgers’ + advection: 1-waves

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Integral curves:

\[ \tilde{u}'(\xi) = 1 \quad \implies \quad \tilde{u}(\xi) = u_* + \xi \]
\[ \tilde{v}'(\xi) = -v(\xi) \quad \implies \quad \tilde{v}(\xi) = v_* e^{-\xi} \]
Burgers’ + advection: 1-waves

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Integral curves:

\[ \tilde{u}'(\xi) = 1 \quad \implies \quad \tilde{u}(\xi) = u^* + \xi \quad \implies \quad \xi = \tilde{u} - u^* \]

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Burgers’ + advection: 1-waves

\[ \lambda^1 = u, \quad r^1 = \left[ \begin{array}{c} 1 \\ -v \end{array} \right], \quad \nabla \lambda^1 \cdot r^1 \equiv 1, \quad \text{genuinely nonlinear} \]

Integral curves:

\[ \tilde{u}'(\xi) = 1 \implies \tilde{u}(\xi) = u_* + \xi \implies \xi = \tilde{u} - u_* \]

\[ \tilde{v}'(\xi) = -v(\xi) \implies \tilde{v}(\xi) = v_* e^{-\xi} \implies \tilde{v} = v_* e^{u_* - \tilde{u}}. \]
$$q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u + 1)v \end{bmatrix}.$$ 

States $q$ and $q_*$ must satisfy Rankine-Hugoniot jump condition:

$$f(q) - f(q_*) = s(q - q_*).$$

First equation gives:

$$\frac{1}{2}(u^2 - u_*^2) = s(u - u_*) = \frac{1}{2}(u + u_*)(u - u_*).$$

One solution:

$$u = u_* \text{ (and jump in } v \text{ arbitrary)} = \text{vertical lines}$$

These are Hugoniot loci for 2-waves.

2-waves are discontinuities in $v$ alone, speed $s = u_* + 1$ (determined from second equation of R-H conditions).
Burgers’ + advection: Hugoniot loci

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Second solution:

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s = s^1 = \frac{1}{2}(u + u_*) \implies \text{shock waves in Burgers’ equation}
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Relation between \(v\) and \(u\) across shock:

Second equation of R-H relation:

\[
(u + 1)v - (u_* + 1)v_* = s(v - v_*) = \frac{1}{2}(u + u_*)(v - v_*)
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Burgers’ + advection: Hugoniot loci

\[
\frac{1}{2}(u^2 - u_*)^2 = s(u - u_*) \implies \frac{1}{2}(u + u_*)(u - u_*) = s(u - u_*).
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\[
\implies v = \left(1 + \frac{1}{2}(u_* - u)\right) v_* \approx e^{u_* - u} v_*
\]

The Hugoniot locus agrees to \(O(|u_* - u|^3)\) with integral curve.
Burgers’ + advection: Phase plane
But note that

\[ v = \left( \frac{1 + \frac{1}{2}(u_* - u)}{1 - \frac{1}{2}(u_* - u)} \right)^{v_*} \to \infty \quad \text{as} \quad u \to u_* - 2 \]
Burgers’ + advection: Riemann solution

To be discussed on the board...

See also the description and codes at

http://www.clawpack.org/links/burgersadv