Today:
- Nonlinear systems of conservation laws
- Shallow water equations
- Characteristics
- Rankine-Hugoniot condition, Hugoniot locus
- Solving Riemann problems

Friday:
- Integral curves, rarefaction waves

Reading: Chapter 13

Shallow water equations

\[ h(x, t) = \text{depth} \]
\[ u(x, t) = \text{velocity (depth averaged, varies only with } x) \]

Conservation of mass and momentum \( hu \) gives system of two equations.

mass flux = \( hu \),
momentum flux = \( (hu)u + p \) where \( p = \text{hydrostatic pressure} \)

\[
\begin{align*}
\frac{h_t + (hu)_x}{(hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x} &= 0
\end{align*}
\]

Jacobian matrix:
\[
f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.
\]

Two-shock Riemann solution for shallow water

Initially \( h_l = h_r = 1, \ u_l = -u_r = 0.5 > 0 \)

Solution at later time:
Two-shock Riemann solution for shallow water

Characteristic curves \( X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)} \)

Slope of characteristic is constant in regions where \( q \) is constant. (Shown for \( g = 1 \) so \( \sqrt{gh} = 1 \) everywhere initially.)

Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.

An isolated shock

If an isolated shock with left and right states \( q_l \) and \( q_r \) is propagating at speed \( s \)

then the Rankine-Hugoniot condition must be satisfied:

\[
  f(q_r) - f(q_l) = s(q_r - q_l)
\]

For a system \( q \in \mathbb{R}^m \) this can only hold for certain pairs \( q_l, q_r \):

For a linear system, \( f(q_r) - f(q_l) = A q_r - A q_l = A(q_r - q_l) \).

So \( q_r - q_l \) must be an eigenvector of \( f'(q) = A \).

\( A \in \mathbb{R}^{m \times m} \Longrightarrow \) there will be \( m \) rays through \( q_l \) in state space in the eigen-directions, and \( q_r \) must lie on one of these.

For a nonlinear system, there will be \( m \) curves through \( q_l \) called the Hugoniot loci.

Hugoniot loci for shallow water

\[
  q = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{bmatrix},
\]

Fix \( q_* = (h_*, u_*) \).

What states \( q \) can be connected to \( q_* \) by an isolated shock?

The Rankine-Hugoniot condition \( s(q - q_*) = f(q) - f(q_*) \) gives:

\[
  s(h_* - h) = h_* u_* - hu,
  s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \frac{1}{2} g(h_*^2 - h^2).
\]

Two equations with 3 unknowns \( (h, u, s) \), so we expect 1-parameter families of solutions.
**Hugoniot loci for shallow water**

Rankine-Hugoniot conditions:

\[
s(h^* - h) = h^*u^* - hu,
\]

\[
s(h^*u^* - hu) = h^*u^*2 - hu^*2 + \frac{1}{2} g(h^*2 - h^2).
\]

For any \( h > 0 \) we can solve for

\[
u(h) = u^* \pm \sqrt{\frac{g}{2} \left( \frac{h^*}{h} - \frac{h}{h^*} \right) (h^* - h)}.
\]

\[s(h) = (h^*u^* - hu)/(h^* - h).
\]

This gives 2 curves in \( h-hu \) space (one for +, one for −).

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**Notes:**

For any \( h > 0 \) we have a possible shock state. Set \( h = h^* + \alpha \), so that \( h = h^* \) at \( \alpha = 0 \), to obtain

\[
hu = h^*u^* + \alpha \left[ u^* \pm \sqrt{gh^* + \frac{1}{2}g\alpha(3 + \alpha/h^*)} \right].
\]

Hence we have

\[
\begin{bmatrix}
h \\
hu
\end{bmatrix} = \begin{bmatrix}
h^* \\
h^*u^*
\end{bmatrix} + \alpha \begin{bmatrix}
1 \\
\sqrt{gh^* + O(\alpha)}
\end{bmatrix}
\]

as \( \alpha \to 0 \).

Close to \( q_* \), the curves are tangent to eigenvectors of \( f'(q_*) \).

Expected since \( f(q) - f(q_*) \approx f'(q_*)(q - q_*) \).

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**Hugoniot loci for one particular \( q_* \)**

States that can be connected to \( q_* \) by a “shock”

Note: Might not satisfy entropy condition.
Hugoniot loci for two different states

“All-shock” Riemann solution:
From $q_l$ along 1-wave locus to $q_m$,
From $q_r$ along 2-wave locus to $q_m$,

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2-shock Riemann solution for shallow water

Given arbitrary states \( q_l \) and \( q_r \), we can solve the Riemann problem with two shocks.

Choose \( q_m \) so that \( q_m \) is on the 1-Hugoniot locus of \( q_l \) and also \( q_m \) is on the 2-Hugoniot locus of \( q_r \).

This requires

\[
u_m = u_r + (h_m - h_r) \sqrt{\frac{g}{2} \left( \frac{1}{h_m} + \frac{1}{h_r} \right)}
\]

and

\[
u_m = u_l - (h_m - h_l) \sqrt{\frac{g}{2} \left( \frac{1}{h_m} + \frac{1}{h_l} \right)}.
\]

Equate and solve single nonlinear equation for \( h_m \).

Hugoniot loci for one particular \( q^* \)

Green curves are contours of \( \lambda^1 \)

Note: Increases in one direction only along blue curve.

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:

Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.
2-shock Riemann solution for shallow water

Colliding with \( u_l = -u_r > 0 \):

Entropy condition: Characteristics should impinge on shock:
- \( \lambda_1 \) should decrease going from \( q_l \) to \( q_m \),
- \( \lambda_2 \) should increase going from \( q_r \) to \( q_m \),

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Notes:

Two-shock Riemann solution for shallow water

Characteristic curves \( X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)} \)

Slope of characteristic is constant in regions where \( q \) is constant. (Shown for \( g = 1 \) so \( \sqrt{gh} = 1 \) everywhere initially.)

Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.

Notes:

2-shock Riemann solution for shallow water

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This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Notes:
Entropy-violating Riemann solution for dam break

**Characteristic curves**

\[ X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)} \]

Slope of characteristic is constant in regions where \( q \) is constant.

Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

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**The Riemann problem**

Dam break problem for shallow water equations

\[ h_t + (hu)_x = 0 \]
\[ (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x = 0 \]