Today:
- Nonlinear systems of conservation laws
- Shallow water equations
- Characteristics
- Rankine-Hugoniot condition, Hugoniot locus
- Solving Riemann problems

Friday:
- Integral curves, rarefaction waves

Reading: Chapter 13
Shallow water equations

\( h(x, t) \) = depth
\( u(x, t) \) = velocity (depth averaged, varies only with \( x \))

Conservation of mass and momentum \( hu \) gives system of two equations.

mass flux = \( hu \),
momentum flux = \( (hu)u + p \) where \( p \) = hydrostatic pressure

\[
\begin{align*}
ht + (hu)_x &= 0 \\
(hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x &= 0
\end{align*}
\]

Jacobian matrix:

\[
f'(q) = \begin{bmatrix}
0 & 1 \\
gh - u^2 & 2u
\end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.
\]
Two-shock Riemann solution for shallow water

Initially $h_l = h_r = 1$, $u_l = -u_r = 0.5 > 0$

Solution at later time:
Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where $q$ is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)

Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.
An isolated shock

If an isolated shock with left and right states $q_l$ and $q_r$ is propagating at speed $s$

then the **Rankine-Hugoniot** condition must be satisfied:

\[
f(q_r) - f(q_l) = s(q_r - q_l)
\]

For a system $q \in \mathbb{R}^m$ this can only hold for certain pairs $q_l, q_r$:

For a **linear system**, $f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l)$.

So $q_r - q_l$ must be an eigenvector of $f'(q) = A$. 
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$A \in \mathbb{R}^{m \times m} \implies$ there will be $m$ rays through $q_l$ in state space in the eigen-directions, and $q_r$ must lie on one of these.
An isolated shock

If an isolated shock with left and right states \( q_l \) and \( q_r \) is propagating at speed \( s \)

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For a **linear system**, \( f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l) \).

So \( q_r - q_l \) must be an eigenvector of \( f'(q) = A \).

\( A \in \mathbb{R}^{m \times m} \) \( \implies \) there will be \( m \) rays through \( q_l \) in state space in the eigen-directions, and \( q_r \) must lie on one of these.

For a **nonlinear system**, there will be \( m \) curves through \( q_l \) called the **Hugoniot loci**.
Hugoniot loci for shallow water

\[ \begin{align*}
q &= \left[ \begin{array}{c}
h \\
h u
\end{array} \right], \\
\mathbf{f}(q) &= \left[ \begin{array}{c}
h u \\
h u^2 + \frac{1}{2}gh^2
\end{array} \right].
\end{align*} \]

Fix \( q^* = (h^*, u^*) \).

What states \( q \) can be connected to \( q^* \) by an isolated shock?

The Rankine-Hugoniot condition \( s(q - q^*) = \mathbf{f}(q) - \mathbf{f}(q^*) \) gives:

\[ \begin{align*}
s(h^* - h) &= h^*u^* - hu, \\
s(h^*u^* - hu) &= h^*u^2 - hu^2 + \frac{1}{2}g(h^2 - h^2).
\end{align*} \]

Two equations with 3 unknowns \((h, u, s)\), so we expect 1-parameter families of solutions.
Hugoniot loci for shallow water

Rankine-Hugoniot conditions:

\[ s(h_\ast - h) = h_\ast u_\ast - hu, \]
\[ s(h_\ast u_\ast - hu) = h_\ast u_\ast^2 - hu^2 + \frac{1}{2}g(h_\ast^2 - h^2). \]

For any \( h > 0 \) we can solve for

\[ u(h) = u_\ast \pm \sqrt{\frac{g}{2}} \left( \frac{h_\ast}{h} - \frac{h}{h_\ast} \right) (h_\ast - h) \]
\[ s(h) = (h_\ast u_\ast - hu)/(h_\ast - h). \]

This gives 2 curves in \( h-hu \) space (one for \( + \), one for \( - \)).
For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_* u_* + \alpha \left[ u_* \pm \sqrt{gh_* + \frac{1}{2}g\alpha(3 + \alpha/h_*)} \right].$$
For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_* u_* + \alpha \left[ u_* \pm \sqrt{gh_* + \frac{1}{2}g\alpha(3 + \alpha/h_*)} \right].$$

Hence we have

$$\begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} h_* \\ h_* u_* \end{bmatrix} + \alpha \left[ u_* \pm \sqrt{gh_* + O(\alpha)} \right]$$

as $\alpha \to 0$.

Close to $q_*$ the curves are tangent to eigenvectors of $f'(q_*)$.

Expected since $f(q) - f(q_*) \approx f'(q_*)(q - q_*)$. 

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Hugoniot loci for one particular $q_*$

States that can be connected to $q_*$ by a “shock”

Note: Might not satisfy entropy condition.
Hugoniot loci for two different states

“All-shock” Riemann solution:

From $q_l$ along 1-wave locus to $q_m$,
From $q_r$ along 2-wave locus to $q_m$, 
All-shock Riemann solution

From $q_l$ along 1-wave locus to $q_m$,
From $q_r$ along 2-wave locus to $q_m$. 

Hugoniot loci in phase plane
All-shock Riemann solution

From $q_l$ along 1-wave locus to $q_m$,
From $q_r$ along 2-wave locus to $q_m$, 
Given arbitrary states $q_l$ and $q_r$, we can solve the Riemann problem with two shocks.

Choose $q_m$ so that $q_m$ is on the 1-Hugoniot locus of $q_l$ and also $q_m$ is on the 2-Hugoniot locus of $q_r$.

This requires

$$u_m = u_r + (h_m - h_r) \sqrt{\frac{g}{2} \left( \frac{1}{h_m} + \frac{1}{h_r} \right)}$$

and

$$u_m = u_l - (h_m - h_l) \sqrt{\frac{g}{2} \left( \frac{1}{h_m} + \frac{1}{h_l} \right)}.$$

Equate and solve single nonlinear equation for $h_m$. 

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AMath 574, February 23, 2011  [FVMHP Sec. 13.7]
Hugoniot loci for one particular $q_*$

Green curves are contours of $\lambda^1$

Note: Increases in one direction only along blue curve.
Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:

Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

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AMath 574, February 23, 2011  [FVMHP Fig. 13.9]
Colliding with $u_l = -u_r > 0$:

![Diagram showing Hugoniot loci and entropy condition](FVMHP Fig. 13.10)
Colliding with $u_l = -u_r > 0$:

Entropy condition: Characteristics should impinge on shock:
- $\lambda^1$ should decrease going from $q_l$ to $q_m$,
- $\lambda^2$ should increase going from $q_r$ to $q_m$,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).
Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where $q$ is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)

Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.
2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$: Entropy condition: Characteristics should impinge on shock:

$\lambda_1$ should decrease going from $q_l$ to $q_m$,
$\lambda_2$ should increase going from $q_r$ to $q_m$,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

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AMath 574, February 23, 2011 [FVMHP Fig. 13.10]
2-shock Riemann solution for shallow water

Colliding with \( u_l = -u_r > 0 \):

**Entropy condition:** Characteristics should impinge on shock:
- \( \lambda^1 \) should decrease going from \( q_l \) to \( q_m \),
- \( \lambda^2 \) should increase going from \( q_r \) to \( q_m \),

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

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AMath 574, February 23, 2011  [FVMHP Fig. 13.10]
Characteristic curves \( X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)} \)

Slope of characteristic is constant in regions where \( q \) is constant.

Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.
The Riemann problem

Dam break problem for shallow water equations

\[ h_t + (hu)_x = 0 \]

\[ (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \]
The Riemann problem

Dam break problem for shallow water equations

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