Each part of each problem is worth 8 points (96 points total).

Please show all of your work and justify all your answers.

1. (a) Show that for any \( x \in \mathbb{C}^m \), \( \|x\|_\infty \leq \|x\|_1 \).

   \[
   \|x\|_\infty = \max_i |x_i| \leq \sum_i |x_i| = \|x\|_1.
   \]

(b) Show that for any \( x \in \mathbb{C}^m \), \( \|x\|_1 \leq m \|x\|_\infty \).

   If \( x_j \) is the element of \( x \) with maximum modulus then

   \[
   \|x\|_1 = \sum_i |x_i| \leq m |x_j| = m \|x\|_\infty.
   \]

(c) Show that bounds of the form

   \[
   c_1 \|A\|_1 \leq \|A\|_\infty \leq c_2 \|A\|_1
   \]

hold for any matrix \( A \in \mathbb{C}^{m \times n} \), where the constants \( c_1 \) and \( c_2 \) depend only on \( m \) and \( n \) (not on the particular matrix). Determine these constants (the best possible, using the bounds from parts (a) and (b)). Hint: Tackle each inequality separately.

   \[
   \|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \frac{\|Ay\|_1}{\|y\|_1} \leq \frac{m \|Ay\|_\infty}{\|y\|_\infty} \leq m \|A\|_\infty
   \]

   where \( y \) is the vector that maximizes the ratio. So \( c_1 = 1/m \).

   \[
   \|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{\|Ay\|_\infty}{\|y\|_\infty} \leq \frac{n \|Ay\|_1}{\|y\|_1} \leq n \|A\|_1
   \]

   where \( y \in \mathbb{C}^n \) is the vector that maximizes the ratio. So \( c_2 = n \). This uses

   \[
   \|y\|_1 \leq n \|y\|_\infty \implies \frac{1}{\|y\|_\infty} \leq \frac{n}{\|y\|_1}
   \]

   since \( y \in \mathbb{C}^n \).

2. Let

   \[
   A = \begin{bmatrix}
   1 & 3 \\
   1 & -3 \\
   1 & 3 \\
   1 & -3
   \end{bmatrix}.
   \]

   (a) Determine the reduced QR factorization of the matrix \( A \).

   Normalizing the first column of \( A \) gives

   \[
   q_1 = a_1/\|a_1\| = \frac{1}{2} \begin{bmatrix}
   1 \\
   1 \\
   1 \\
   1
   \end{bmatrix}.
   \]
Since $a_1^T a_2 = 0$ the columns are already orthogonal so we only need to normalize the second column to get $q_2$ by dividing by $\|a_2\| = 6$.

We find that

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$ (b) Using any method you wish, solve the least squares problem $Ax = b$ for

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$ You can form the normal equations $A^T Ax = A^T b$ or solve the system $Rx = Q^T b$. Either way you need only solve a diagonal system of two equations for

$$x = \begin{bmatrix} 3/4 \\ 1/12 \end{bmatrix}.$$

3. For this problem, let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector, and let $Q \in \mathbb{C}^{m \times n}$ with $n < m$ and $Q^*Q = I_{n \times n}$.

(a) Is $Q$ unitary? (Justify your answer.)

No, only a square matrix can be unitary. $Q^*Q = I$ but $QQ^* \neq I$.

(b) Show that $\|Q\|_2 = 1$.

For any $x$, $\|Qx\|^2 = x^*Q^*Qx = x^*x = \|x\|^2$ and hence

$$\frac{\|Qx\|}{\|x\|} = 1$$

for all $x$ and so the norm is 1.

(c) Show that $\|P\|_2 \geq 1$.

Any projector satisfies $P^2 = P$. Choose $x$ so that $y = Px \neq 0$. Then $P(Px) = P^2x = Px$ and hence $Py = y$ and $\|Py\|/\|y\| = 1$. The matrix norm is the maximum of this ratio and must be at least this large.

Note the problem stated that $P$ is a nonzero projector. The zero matrix is a projector but has norm 0. The proof above would fail since there is no $x$ for which $Px \neq 0$ in this case.

(d) Suppose $P$ is an orthogonal projector (recall that this means $P = P^*$, not $P^*P = I$). Show that $\|P\|_2 = 1$. Hint: Write $P$ in terms of a matrix with the properties of $Q$.

Any orthogonal projector can be written as $P = QQ^*$ where the columns of $Q$ are an orthogonal basis for the range of $P$.

So $\|P\| \leq \|Q\|\|Q^*\|$. We know $\|Q\| \leq 1$ and also $\|A^*\| = \|A\|$ in the 2-norm for any $A$, so we also have $\|Q^*\| = 1$. 2
Note that in other norms it is not always true that $\|A^*\| = \|A\|$, since, for example $\|A^*\|_1 = \|A\|_\infty$. For the 2-norm you can verify that from the fact that

$$A = U\Sigma V^* \implies A^* = V\Sigma^*,$$

so both matrices have the same $\sigma_1 = \|A\|_2$.

For an orthogonal projector, you could also note that $P = QIQ^*$ is the SVD of $P$ and so $\sigma_1 = 1$.

4. Let $A = e_1e_2^* + 2e_2e_3^* = e_1e_2^T + 2e_2e_3^T \in \mathbb{R}^{3 \times 3}$, where $e_1$, $e_2$, and $e_3$ are the unit vectors in $\mathbb{R}^3$, i.e. the three columns of the $3 \times 3$ identity matrix.

(a) What is the rank of $A$?

The rank is 2. There are several ways to justify this. For example by noting that the singular values are 2, 1, 0 and only 2 are nonzero.

(b) Determine the full SVD of the matrix $A$. Hint: $U$ and $V$ will be permutation matrices. Another hint: What are the three singular values of $A$?

Write $A = 2e_2e_3^* + e_1e_2^* + 0e_3e_1^*$ to see that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Remember that the proper SVD form requires the $\sigma_i$ in decreasing order.

(c) Determine the projection matrix $P$ that orthogonally projects any vector in $\mathbb{R}^3$ onto the range of $A$.

$P = \hat{U}\hat{U}^*$ where $\hat{U}$ is the reduced $U$ with only the columns corresponding to nonzero singular values. These columns form an orthogonal basis for the range of $A$.

$$\hat{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \implies P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

You could also deduce this from the original form of $A$ given: $Ax$ is a linear combination of $e_1$ and $e_2$ for any $x$, so $P$ must project onto the $x_1-x_2$ plane.

Note that if you use the full $U$ instead of $\hat{U}$ you would get $P = I$, which can’t be right since it has rank 3 instead of 2. Since $U$ is unitary its columns span all of $\mathbb{R}^3$. 

3