Kinetic Energy

\[ T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \]

\[ T = \frac{1}{2} \sum_{k=1}^{n} m_k V_k^2 \]

where \[ V_k^2 = \dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2 \]

\[ x_k = x_k(q_1, q_2, \ldots, q_n), \quad y_k = y_k(q_1, q_2, \ldots, q_n) \]

\[ \frac{dx_k}{dt} = \dot{x}_k = \frac{\partial x_k}{\partial q_i} \dot{q}_i + \frac{\partial x_k}{\partial q_j} \dot{q}_j + \ldots \]

\[ V_k^2 = \left( \frac{\partial x_k}{\partial q_i} \right)^2 + \left( \frac{\partial y_k}{\partial q_i} \right)^2 + \left( \frac{\partial z_k}{\partial q_i} \right)^2 \dot{q}_i^2 + \left[ \text{same wrt } q_j \right] \dot{q}_j^2 + \ldots \]

\[ V_k^2 = \left[ \left( \frac{\partial x_k}{\partial q_i} \right)^2 \left( \frac{\partial y_k}{\partial q_i} \right)^2 \left( \frac{\partial z_k}{\partial q_i} \right)^2 \right] \dot{q}_i^2 + \ldots \]

\[ T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j + O(q^3) \]

where \[ m_{ij} = \sum_{k=1}^{m_k} \left[ \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} + \frac{\partial y_k}{\partial q_i} \frac{\partial y_k}{\partial q_j} + \frac{\partial z_k}{\partial q_i} \frac{\partial z_k}{\partial q_j} \right] \]

Expand \( m_{ij} \) about \( q_0 \)

\[ m_{ij} = m_{ij} \bigg|_{q_0} + \frac{\partial m_{ij}}{\partial q_l} \bigg|_{q_0} q_l + \frac{1}{2} \sum_{p=1}^{n} \frac{\partial^2 m_{ij}}{\partial q_l \partial q_p} \bigg|_{q_0} q_l q_p + \ldots \]

If we want to keep \( T \) to second order in \( q \) as we have for \( V \) we must neglect all terms in the expansion of higher order than \( O(1) \)
Multi Degree of Freedom

Potential Energy:

\[ V = V(q_1, q_2, \ldots, q_n) \]

Select the reference potential \( V_0 = 0 \) and assume that this represents the equilibrium configuration.

Then

\[ V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j \]

where \( k_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \)

Note that \( k_{ij} = k_{ji} \) symmetric.

If \( k_{ij} \neq 0 \) when \( i \neq j \) then the system is said to be statically coupled.
\[
\Rightarrow m_{ij} = m_{ij}|_0 = \text{const} \\
\]
Again note that \( m_{ij} = m_{ji} \)

so

\[
\frac{1}{2} \sum_{i=1}^{n} m_{ij} q_i q_j \\
\]

if \( m_{ij} \rightarrow 0 \) when \( i = j \) the system is said to be dynamically coupled.

Define

\[
k = \text{stiffness matrix } \quad k = \{k_{ij}\} \\
M = \text{mass matrix } \quad M = \{m_{ij}\} \\
q = \text{vector of elements } q_1, q_2, \ldots \\
q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}
We note that the potential energy can be expressed as

$$V = \frac{1}{2} q^T K q \geq 0 \quad \forall q$$

Also

$$T = \frac{1}{2} q^T M q \geq 0 \quad \forall q^*$$

(Both $K$ and $M$ are positive definite)

Substitute into L.E.

$$\frac{d}{dt} \frac{\partial (T-V)}{\partial q_i} - \frac{\partial}{\partial q_i} (T-V) = 0$$

Show

$$M \ddot{q}^0 + K q^0 = 0$$
Solution of $M\ddot{x} + Kx = 0$

Normal Mode Approach

Let $x = Xe^{i\omega t}$

Substitute into $D = 0$:

$$(-\omega^2 M + K)x e^{i\omega t} = 0$$

$$(M^{-1}K - \omega^2 I)X = 0$$  \quad \text{(eigenvalue problem)}

For nontrivial solutions:

$$\text{det}(M^{-1}K - \omega^2 I) = 0 \quad \text{characteristic equation}$$

The characteristic equation will be an $n^{th}$ order polynomial in $\omega^2$ where $n$ is the system.

$(M^{-1}K = H : \text{The Dynamical Matrix})$

Assume that all of the roots of $\text{det}(M^{-1}K - \omega^2 I)$ are distinct.

Then for each root (eigenvalue), $\omega_r$, there will be a corresponding mode vector (eigenvector) $\vec{X}_r$.

$$(M^{-1}K - \omega_r^2 I)\vec{X}_r = 0$$

Solve for $\vec{X}_r$ to get mode shape.