5.44 (cont.) The k₁ spring isn't deformed during the motion

mode shapes

The next mode will look like

\[ \beta_2 = 3.157427 \] (higher than \( \beta_1 \) due to the

\[ \omega_j = 25.785 \text{ rad/s} \] (k₁ participation)

5.45

Eigen function:

\[ \bar{s}(x) = a_1 \cos \beta x + b_1 \sin \beta x \]

\[ \bar{\eta}(y) = a_2 \cos \beta y + b_2 \sin \beta y \]

Since the beams are identical the same \( \beta \)'s are used for each beam.

B.C.: \[ \bar{s}(0) = 0 \Rightarrow a_1 = 0 \] \( (\bar{s}(x) = b_1 \sin \beta x) \)

B.C.: \[ \bar{\eta}(0) = 0 \Rightarrow a_2 \cos \beta + b_2 \sin \beta = 0 \]

B.C.: \[ (\bar{\eta}(0) - \bar{s}(0))k₁ - EA \bar{s}(0) = 0 \Rightarrow (a_2 - b_1 \sin \beta)k₁ - EA(1 - b_2 \cos \beta) = 0 \]

B.C.: \[ (\bar{s}(0) - \bar{\eta}(0))k₁ + EA \bar{\eta}(0) = 0 \Rightarrow (b_1 \sin \beta - a_2)k₁ + EA(b_2 - \beta) = 0 \]

The first \( \beta \) is \( \beta_1 = \frac{\pi}{2} \). This corresponds to each bar undergoing fixed-free vibrations. Both ends move in phase and thus the spring \( k₁ \) isn't deformed.

For \( \beta_2 \) we have symmetric motions towards \& away from each other. The \( \beta \) is equivalent to the one found for the illustrated "half" system.

\[ \beta_2 = 1.602 \] (higher than \( \beta_1 \) since the spring is now involved)
5.51

For the clamped-clamped case we have
\[ \bar{w}_x(0) = \bar{w}_x(1) = 0 \text{ and } \bar{w}(0) = \bar{w}(1) = 0. \]
We know \( w(x) = a \cos \beta x + b \sin \beta x + c \cosh \beta x + d \sinh \beta x \)
\[ \bar{w}(0) = 0 \Rightarrow a + c = 0 \Rightarrow a = -c \]
\[ \bar{w}_x(0) = 0 \Rightarrow b + d = 0 \Rightarrow b = -d \]

So \[ \bar{w}(x) = a (\cos \beta x - \cosh \beta x) + b (\sin \beta x - \sinh \beta x) \]
\[ \bar{w}(1) = 0 \Rightarrow a (\cos \beta x - \cosh \beta x) + b (\sin \beta x - \sinh \beta x) = 0 \]
\[ \bar{w}_x(1) = 0 \Rightarrow a (-\sin \beta x - \sinh \beta x) + b (\cos \beta x - \cosh \beta x) = 0 \]
\[ \begin{bmatrix} \cos \beta x - \cosh \beta x & \sin \beta x - \sinh \beta x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Taking the determinant: \[ (\cos \beta x - \cosh \beta x) (\cos \beta x - \cosh \beta x) + (\sin \beta x - \sinh \beta x) (\sin \beta x - \sinh \beta x) = 0 \]
\[ \cos^2 \beta x + \sinh^2 \beta x - 2 \cos \beta x \cosh \beta x + \sinh^2 \beta x = 0 \]
\[ 2 - 2 \cos \beta x \cosh \beta x = 0, \quad \boxed{\cos \beta x \cosh \beta x = 1} \]

For the free-free case we have \[ \bar{w}_{xx}(0) = \bar{w}_{xx}(1) = 0 \]
\[ \bar{w}_{xx}(0) = 0 \Rightarrow -b + d = 0 \text{ or } b = d \]
\[ \bar{w}_{xx}(1) = 0 \Rightarrow -a + c = 0 \text{ or } a = c \]

\[ \bar{w}(x) = a (\cos \beta x + \cosh \beta x) + b (\sin \beta x + \sinh \beta x) \]
\[ \bar{w}_{xx}(0) = 0 \Rightarrow a (-\cos \beta x + \cosh \beta x) + b (-\sin \beta x + \sinh \beta x) = 0 \]
\[ \bar{w}_{xx}(1) = 0 \Rightarrow a (\sin \beta x + \sinh \beta x) + b (-\cos \beta x + \cosh \beta x) = 0 \]
\[ \begin{bmatrix} \cos \beta x + \cosh \beta x & -\sin \beta x + \sinh \beta x \\ \sin \beta x + \sinh \beta x & -\cos \beta x + \cosh \beta x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Determinant: \[ (-\cos \beta x + \cosh \beta x)^2 - (\sin \beta x + \sinh \beta x) (-\sin \beta x + \sinh \beta x) = 0 \]
\[ \cos^2 \beta x - 2 \cos \beta x \cosh \beta x + \cosh^2 \beta x + \sin^2 \beta x - \sinh^2 \beta x = 0 \]
\[ 2 - 2 \cos \beta x \cosh \beta x = 0 \]
\[ \boxed{\cos \beta x \cosh \beta x = 1} \]

Both cases have the same \( \beta \) equation and therefore the same \( \beta_n \)'s.
PROBLEM 5.67

\[ Ty_{xx} - p^2 y = 0, \quad y(0) = 0, \quad y_x(0) = 0 \]

\text{ASSUME } y = y \sin(\omega t)

\[ Ty_{xx} + \frac{p \omega^2}{T} y = 0 \]
\[ Ty_{xx} + \frac{p \omega^2}{T} y = 0 \]
\[ Ty_{xx} + \beta^2 y = 0, \quad \beta^2 = \frac{p \omega^2}{T} \]

\text{SOLUTION: } y(x) = a_1 \cos(\beta x) + a_2 \sin(\beta x)

\text{APPLYING } y(0) = 0 \text{ GIVES } a_1 = 0

\[ y(x) = a_2 \sin(\beta x) \]

\text{APPLYING } y_x(0) = 0 \text{ GIVES }

\[ \cos(\beta 0) = 0 \Rightarrow \beta_n d = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \ldots \]

\[ \beta_n = \frac{(2n-1)\pi}{2d} \]

\text{EIGENFUNCTION IS } \sin \left( \frac{(2n-1)\pi x}{d} \right)

\[ \int_0^d \sin \left( \frac{(2n-1)\pi x}{d} \right) \sin \left( \frac{(2m-1)\pi x}{d} \right) dx = 0, \quad n \neq m \]

\text{EIGENFUNCTION SATISFIES ORTHOGONALITY}
5.40

\( I_0 = 1.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2 \)

\( \bar{\theta}(x) = a_1 \cos \beta x + a_2 \sin \beta x \)

\( \bar{\theta}(0) = 0 \Rightarrow a_1 = 0 \)

At \( x = L \) we have the B.C.: \( I_0 \ddot{\theta} = -GJ \bar{\theta}_x \)

\( -\omega^2 I_0 \bar{\theta}(L) = -GJ \bar{\theta}_x(L) \)

\( \omega^2 = \frac{\beta^2}{p} \) so this becomes

\[ -\beta \frac{E_0}{\rho} \sin \beta L + J \beta \cos \beta L = 0 \]

\[ -\beta \frac{E_0}{\rho} \sin \beta L + J \rho \cos \beta L = 0 \quad (1) \]

Adding \( I_0 \) to the shaft reduces the fixed-free natural frequencies. For the fixed-free case the boundary condition \( \bar{\theta}_x(L) = 0 \) gives \( \beta \) solutions of

\[ \beta_n = \frac{(2n-1)\pi}{L} \]

The first natural frequency \( \beta \) is thus \( \beta_1 = \frac{\pi}{2L} \). The problem is therefore to find an \( I_0 \) such that the new \( \beta_1 \) is 90% of the original, or

\[ 0.9 \left( \frac{\pi}{2L} \right) = 0.9 \left( \frac{\pi}{2(2)} \right) = 0.7069 \]

Calculate \( J \):

\[ J = \frac{\pi}{2} (0.011^4 - 0.01^4) = 7.29 \times 10^{-9} \]

\( (1) \Rightarrow - (0.7069) I_0 \sin \left( 0.7069 \cdot (2) \right) + (7.29 \times 10^{-9})(6.7 \times 10^8) \cos (0.7069 \cdot (2)) = 0 \)
\[ T = 100, \quad \rho = 0.002 \]

\[ x = 0 : \quad \overline{y}(\omega) = 0 \]

\[ x = 1 : \quad k \overline{y}(\omega) - Tw \overline{y}_x(\omega) = 0 \]

\[ \overline{y}(x) = a_1 \sin \beta x + a_2 \cos \beta x \]

\[ \overline{y}(\omega) = 0 \Rightarrow a_2 = 0 \]

B.C. at \( x = 1 : \quad ka_1 \sin \beta l + T \beta a_1 \cos \beta l = 0 \]

\[ k \sin \beta l = -T \beta \cos \beta l \]

WITH \( \lambda = 1, \quad T = 100 \) THIS IS

\[ k \sin \beta + 100 \beta \cos \beta = 0 \]

THE FIRST THREE \( \beta \)S, PLOTTED AS A FUNCTION OF \( k \), ARE SHOWN BELOW
The general solution for a tensioned string problem is found from the equation of motion:

\[ Ty_{xx} - \rho y = 0 \]

If \( y(x,t) = \bar{y}(x) \cos(\omega t + \phi) \) we have

\[ Ty_{xx} + \omega^2 \rho \bar{y} = 0, \quad y_{xx} + \omega^2 \bar{y} = 0 \]

Set \( \beta^2 = \omega^2 \rho \) to obtain \( \bar{y}_{xx} + \beta^2 \bar{y} = 0 \)

general solution to this is \( \bar{y}(x) = a \sin \beta x + b \cos \beta x \)

For the fully restrained case \( \bar{g}(0) = \bar{g}(l) = 0 \)

\( y(0) = 0 \Rightarrow y(x) = \alpha \sin \beta x \)

\( y(l) = 0 \Rightarrow a \sin \beta l = 0 \Rightarrow \beta = \frac{n \pi}{l} \)

For the unrestrained case we have \( \bar{y}(0) = \bar{y}(l) = 0 \)

This can be seen from a force balance:

Assume for a moment that there is a mass \( \varepsilon \) at \( x = 0 \). We have

\[ \varepsilon \ddot{y} = T \sin \theta \]

Since \( \sin \theta = \lim_{x \to 0} \frac{\ddot{y}}{\dot{y}} \) we have

\[ \varepsilon \ddot{y} = T y_{xx} \]

But actually there is no mass \( \varepsilon \) so we have \( 0 = T y_{xx} \)

This can only be satisfied if \( y_{xx} = 0 \). This is only satisfied if \( y_{xx} = 0 \).

Thus we see that the \( \beta \)'s for the two cases are the same but the eigenfunctions are different (\( \sin \beta x \) for restrained and \( \cos \beta x \) for unrestrained).