Lateral Vibrations of a Beam

\[ W(x) = \text{load/length} \]

\[ M(x) = \int M \, dx \]

1. \[
\varepsilon_{F_y} = \frac{\partial^2 y}{\partial t^2} \Rightarrow W \cdot \delta x + dV = \rho A \, dx \Rightarrow W = -\frac{\partial V}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} \]

2. \[
\Sigma M_{max} = 0 \Rightarrow V \cdot \frac{\partial^2 y}{\partial x^2} + (V + dV) \frac{\partial^2 y}{\partial x^2} = -\frac{\partial M}{\partial x} \Rightarrow V(x,t) = -\frac{\partial M}{\partial x} \]

3. \[
\Rightarrow W(x,t) = \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 y}{\partial t^2} \]

Beam theory: plane sections remain plane

\[ \varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \cdot \hat{y} \right) = \frac{\partial^2 y}{\partial x^2} \cdot \hat{y} \]

Moment: \[
M = \int \sigma_x \hat{y} \, dA = \int \frac{E}{A} \frac{\partial^2 y}{\partial x^2} \hat{y}^2 \, dA = E \frac{\partial^2 y}{\partial x^2} \int \hat{y}^2 \, dA = EI \frac{\partial^2 y}{\partial x^2} \]

4. \[
\Rightarrow W(x,t) = \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 y}{\partial t^2} \]

Equation of motion:

\[
\left[ \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) = W \right]
\]
If $EI$ is constant and $w=0$

$$EI \frac{d^4y}{dx^4} + \rho A \frac{d^2y}{dt^2} = 0$$

Solution: Use separately of variables

Let $y(x,t) = X(x)T(t)$

$$\frac{d^4X}{dx^4} \cdot T(t) + \frac{\rho A}{EI} \frac{d^2T}{dt^2} \cdot X(x) = 0$$

$$- \frac{T(t)}{\frac{d^2T}{dt^2}} = \frac{\rho A}{EI} \frac{X(x)}{\frac{d^4X}{dx^4}}$$

$$\frac{T(t)}{\frac{d^2T}{dt^2}} = g(x)$$

$\frac{T(t)}{\frac{d^2T}{dt^2}} = g(x)$ \implies $\frac{T(t)}{\frac{d^2T}{dt^2}} = g(x) =$ constant

call this constant $\frac{1}{w^2}$

$$\implies \begin{cases} \frac{d^2T}{dt^2} + w^2 T(t) = 0 \implies T(t) = E\sin \omega t + F\cos \omega t \\ \frac{d^4X}{dx^4} - \beta^4 X(x) = 0 \end{cases} \text{ where } \beta^4 = \frac{\rho A w^2}{EI}$$

Let $X(x) = e^{\lambda x}$ \hspace{2cm} subst into (2) \implies $\lambda = \pm \beta, \pm i \beta$

so $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 e^{i \beta x} + c_4 e^{-i \beta x}$

or $X(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x$

where $\sinh \beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}, \cosh \frac{e^{\beta x} + e^{-\beta x}}{2}$
We have 6 unknowns
But we also have boundary conditions and 2 initial conditions.

Pinned-Pinned Beam

\[ y(0,t) = 0 \]
\[ y(l,t) = 0 \]
\[ w(x,t) = \frac{\partial^2 y}{\partial x^2} \bigg|_{x=0} = 0 \]
\[ w(x,t) = \frac{\partial^2 y}{\partial x^2} \bigg|_{x=l} = 0 \]

where

\[ \frac{\partial^2 y}{\partial x^2} = T(x) \left[ -A\beta^2 \sin \beta x - B\beta^2 \cos \beta x + C_1 \beta^2 \sinh \beta x + D\beta^2 \cosh \beta x \right] \]

at \( x = 0 \)

\[ y(0,t) = 0 \implies B + D = 0 \]
\[ \frac{\partial^2 y}{\partial x^2} \bigg|_{x=0} = -B + D = 0 \]

\[ \overline{B + D = 0} \]

at \( x = l \)

\[ y(l,t) = 0 = A\sin pl + C \sinh pl \]
\[ \frac{\partial^2 y}{\partial x^2} \bigg|_{x=l} = 0 = A\sin pl + C \sinh pl \]

\[ \implies \begin{cases} A\sin pl = 0 \\ C \sinh pl = 0 \end{cases} \]
But \( \sin p l \geq 0 \) \( \forall p l \geq 0 \)

\[ \Rightarrow \quad \sin p l = 0 \]

and \( A \sin p l = 0 \) \( \Rightarrow \)

\( A = 0 \) \( \Rightarrow \) trivial solution

or

\( \sin p l = 0 \) \( \Rightarrow \) \( \beta_n \cdot l = n \cdot \pi \) \( \Rightarrow \) \( \beta_n = \frac{n \pi}{l} \)

\[ \omega_n = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho}} \quad (\text{since} \quad \beta_n = \frac{n \pi}{l}) \]

natural frequency of the beam

and since \( \delta(x) = A \sin p x + B \cos p x + C \sinh p x + D \cosh p x \)

we get

\[ \delta(x) = A_n \sin \beta_n x \quad n^{th} \text{ mode shape} \]

or

\[ \delta(x) = A_n \sin \frac{n \pi}{l} x \]

and the displacement of the beam is:

\[ y = \sum_{s=0}^{S} \frac{\delta_s}{s!} \tau_s = \sum_{s=0}^{S} \sin \frac{n \pi}{l} x (E \sin \omega_s t + F \cos \omega_s t) \]

\( E \) and \( F \) are determined the same way as for the bar, i.e. given the initial shape and velocity, \( \delta(x, 0) = H(x) \) and \( \dot{\delta}(x, 0) = G(x) \) multiply each side by \( \sin \beta_n x \) and integrate over the length.
Equation of motion

\[ \text{EI} \frac{d^4 y}{dx^4} + \rho A \frac{d^2 y}{dt^2} = W(x,t) \]

Let \( y(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot X_n(t) \)

4 modes

Substitute into DEQ:

\[ \sum_{n=1}^{\infty} \int T_n(t) \left[ \text{EI} \frac{d^4 X_n}{dx^4} + \rho A \frac{d^2 X_n}{dt^2} \right] = W(x,t) \]

But from the unforced problem we know:

\[ \frac{d^4 X_n}{dx^4} - \beta_n^4 X_n = 0 \Rightarrow \frac{d^4 X_n}{dx^4} = \beta_n^4 X_n \]

\[ \Rightarrow \frac{d^4 X_n}{dx^4} = \frac{\rho A}{\text{EI}} w_n^2 X_n \]

so equation 1 becomes

\[ \sum_{n=1}^{\infty} \rho A X_n \left( \frac{\infty}{T_n + \omega_n^2 T_n} \right) = W(x,t) \]

Multiply by \( X_m \) and integrate over the length

\[ \sum_{n=1}^{\infty} \left( \frac{\infty}{T_n + \omega_n^2 T_n} \right) \int_0^L \rho A X_n(x) \cdot \overline{X_m(x)} \, dx = \int_0^L \overline{X_m(x)} \cdot W(x,t) \, dx \]

\[ = 0 \quad n \neq m \]

\[ \neq 0 \quad n = m \]
\[
T_m = \frac{\int_0^\infty X_m(x) \cdot W(x,t) \, dx}{\int_0^\infty \rho A X_m^2(x) \, dx} = f_m(t), \ m = 1, 2, \ldots
\]

\[
T_m = A_m \sin \omega_m t + B_m \cos \omega_m t + \frac{1}{\rho A \omega_m} \int_0^t f(r) \sin \omega_m (t-r) \, dr
\]

The \( A \)'s and \( B \)'s are determined by the initial conditions.

So the motion of the beam becomes:

\[
y(x,t) = \sum_{n=1}^{\infty} \left[ A_n \sin \omega_n t + B_n \cos \omega_n t + \frac{1}{\rho A \omega_n} \int_0^t f(r) \sin \omega_n (t-r) \, dr \right] X_n
\]