Often a physical fluid state is unstable to small perturbations.

E.g., solar heating of air near the ground.

\[ T_2 > T_1, \ p_2 < p_1 \]

Small perturbation to interface:

\[ \rightarrow p^- \leftarrow p^+ \]

Pressure perturbations cause anomaly to grow

Thermals
Stratified shear flows also unstable something

\[ \text{\( u \)} \]

\[ \text{\( \epsilon = \text{const.} \)} \]

initially

\[ \Rightarrow \]

\[ \text{\( \Rightarrow \)} \]

\[ \text{\( \Rightarrow \)} \]

turbulence
Consider 2-layer flow

irrotational

inviscid

barotropic in each layer

- can use methods of surface gravity waves in each layer and match pressure at interface

eg. in layer 1

\[ \bar{u}_1 = (U_1 + u_1, 0, w_1) = \nabla \bar{\phi}_1 \]

small wave perturbations \( \ll U_1 \)

\[ \Rightarrow \quad \bar{\phi}_1 = U_1 x + \phi_1(x, z, t) \]

and \( \nabla^2 \bar{\phi}_1 = \nabla^2 \phi_1 = 0 \)

similar for layer 2
KBC: \( \psi_1 \to 0 \ as \ z \to \infty, \ \psi_1 \to 0 \ as \ z \to -\infty \)

KBC-I:

\[ \frac{\partial \psi_1}{\partial t} = \frac{D\psi_1}{Dt} \quad \text{at} \quad z = \eta \]

\[ \exp(\psi_1) = \psi_1 + \psi_1 \frac{\partial \eta}{\partial x} \]

at \( z = 0 \)

and

\[ \frac{\partial \psi_2}{\partial z} = \frac{\partial \eta}{\partial t} + u_2 \frac{\partial \eta}{\partial x} \]

Dynamic BC: use unstably baroclinic in each layer at \( z = \eta \)

\[ \left[ \frac{d\bar{\psi}_1}{dt} + \frac{1}{2} (\bar{u}_1 \cdot \bar{u}_1) + \frac{\bar{p}_1}{\bar{\rho}_1} + g \bar{z} = F_1(t) \right]_{z = \eta} \]

Define \( \bar{p}_1 = P_1(z) + \bar{p}_1(x, z, t) \)

Similar for layer 2
Note, for undisturbed state

\[ \frac{1}{2} U_1^2 + \frac{P_1}{e_1} = F_1 \]

and

\[ \frac{1}{2} U_2^2 + \frac{P_2}{e_2} = F_2 \]

The linearized form of the far BC is

\[ \left[ \frac{\partial \Phi}{\partial z} + \frac{1}{2} U_1 U_1 + U_1 U_1 + \frac{P_1}{e_1} + \frac{P_1}{e_1} + g \eta \right] \bigg|_{z=0} = 0 \]

and similar for layer 2

then equating the pressure at \( z = 0 \)

+ removing the background \( \Phi \) give

\[ p_1 \left( \frac{\partial \Phi}{\partial t} + U_1 \frac{\partial \Phi}{\partial x} + g \eta \right) = p_2 \left( \frac{\partial \Phi}{\partial t} + U_2 \frac{\partial \Phi}{\partial x} + g \eta \right) \]

at \( z = 0 \)

DBC-I
Then we search for solutions of the form

\[ \varphi = \hat{\Phi}(x) e^{ik(x-ct)} \]

It is implicit that we take the real part.

Note: if \( c \) is complex, \( c = c_R + ic_I \)

Thus

\[ \text{Re} \left\{ e^{ik(x-ct)} \right\} = \text{Re} \left\{ e^{-kC_I t} e^{ik(x-c_R t)} \right\} \]

\[ = e^{-kC_I t} \cos k(x-c_R t) \]

the usual phase propagation

with \( C_P = c_R \)

A growing mode for \( c_I > 0 \)

\[ \Rightarrow \text{Instability} \]
continuing with solution, using
\[ \eta = \eta_0 e^{ik(x-ct)} \]

then using \( \nabla^2 \Phi_1 = 0 \)

\[ -k^2 \Phi_1 + \Phi_1, zz = 0 \]

unknown constant

applying \( kBC - \infty \)

\[ \Phi_1 = A e^{-kz} \]

and \( kBC - I \)

\[ \frac{\partial \Phi_1}{\partial z} = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \bigg|_{z=0} \]

\[ A = -i(U_1 - c) \eta_0 \]

so

\[ \Phi_1 = -i(U_1 - c) \eta_0 e^{-kz} e^{ik(x-ct)} \]

and similarly

\[ \Phi_2 = i(U_2 - c) \eta_0 e^{kz} e^{ik(x-ct)} \]
Plugging these into DBC - I yields

\[ p_1 k (U_1 - c)^2 + p_1 g = -p_2 k (U_2 - c)^2 + p_2 g \]

and this quadratic may be solved for \( c \)

\[
c = \frac{p_2 U_2 + p_1 U_1}{p_2 + p_1} \pm \sqrt{\frac{\varrho \left( \frac{p_2 - p_1}{p_2 + p_1} \right) - p_1 (c \left( \frac{U_2 - U_1}{p_2 + p_1} \right)^2)}{k}}
\]

Simplifying this, for \( U_1 = U_0 + \frac{\Delta U}{2} \), \( U_2 = U_0 - \frac{\Delta U}{2} \)

\[ p_1 = p_0 - \frac{\Delta p}{2} \quad p_2 = p_0 + \frac{\Delta p}{2} \quad \Delta p \ll \rho_0 \]

\[ c = U_0 \pm \sqrt{\frac{\varrho' - \left( \frac{\Delta U}{2} \right)^2}{k}} \quad \text{where} \quad \varrho' = \frac{\varrho \Delta p}{2 \rho_0} \quad \text{"reduced grant"}
\]

adduction by mean flow for \( \Delta U = 0 \rightarrow c = \sqrt{\frac{\varrho'}{k}} \), like \( \sqrt{\frac{1}{k}} \), but much slower!

first imaginary root when \( \varrho' = \left( \frac{\Delta U}{2} \right)^2 \rightarrow \text{growing instability}

\[ k_{crit} = \frac{\varrho'}{(\Delta U/2)^2} \]
There is always an unstable wavelength

\[ |\Delta u| > 0 \]

Result for instability of stratified shear flows

with \( u = U(z) \) and \( p = P(z) \) is that they are unstable for

\[ R_i \sim \text{Richardson } \sim \frac{-\frac{\partial U}{\partial z}}{\frac{1}{80} \frac{\partial^2 P}{\partial z^2}} < \frac{1}{4} \quad (\ast) \]

In the K-H problem, the interface will mix until \((\ast)\) is satisfied, a thickness of about \( k^{-1} \).

\[ \text{should be } k_{\text{crit}}^{-1} \]