DEEP WATER WAVES

Recall shallow water waves:

\[ \eta = \eta_0 \cos(\kx - \omega t) \]

pattern moves at phase speed \( c_p = \frac{\omega}{k} = \sqrt{gH} \) \( \to \) to the right

Hydrostatic because \( H \ll 1 \Rightarrow -\frac{1}{\rho} \frac{\partial p}{\partial z} = -g \eta \]

and so \( \frac{\partial \eta}{\partial z} = 0 \) (note \( [u] \ll c_p \))

What if \( H \) is not \( \ll 1 \) \( \Rightarrow \) non-hydrostatic

We find:

- \( u = u(\z) \)
- \( c_p < \sqrt{gH} \)
- Waves are "dispersive"

\( \Rightarrow \) speed depends on wavelength

(Shorter \( \lambda \to \) slower)
Physical Aspects

\[ p = \rho \text{atm} \]
\[ \eta(x,t) \]
\[ z = 0 \]
\[ z = -H \]

\( p = \text{const.} \)
invicid
\[ \frac{\partial \eta}{\partial x} = 0 \]

irrotational at \( t = 0 \) \( \Rightarrow \) irrotational forever (KCT)

\[ \mathbf{u} = \nabla \phi \quad (u_x, u_z) = \phi_x, \phi_z \]

also incompressible \( \Rightarrow \)

\[ \nabla^2 \phi = 0 \quad \text{Laplace eqn.} \quad (\phi_{xx} + \phi_{zz} = 0) \]

Kinematic Boundary Conditions
Bottom:

\[ w(\zeta = H) = 0 \Rightarrow \frac{\partial \eta}{\partial z} (x, -H, t) = 0 \quad \text{KBC-B} \]

Surface:

- a fluid parcel on the surface
- stays on the surface

\[ \frac{\partial \eta}{\partial t} = w_x(x, \eta) \quad \text{KBC-S} \]

Tani's series expansion

\[ \eta + u \eta_x = \omega(x, 0, t) + \eta \frac{\partial w}{\partial z} (x, 0, t) \]

drop for
\[ uw \ll \omega \]
\[ \frac{u}{c} \ll 1 \]

drop if vertical penetration of wave signal is \( \gg \eta \)
Dynamic Boundary Condition

\[ \nabla \cdot \text{man} \begin{pmatrix} \frac{\partial u}{\partial t} + \nabla \left( \frac{1}{2} u \cdot u \right) + \frac{1}{2} \frac{\partial \mathbf{u}}{\partial t} u = -\frac{1}{\epsilon} \nabla \psi - \nabla (\psi z) \end{pmatrix} \]

\[ \frac{\partial}{\partial t} \nabla \phi = \nabla \left( \psi \right) \]

\[ \nabla \left( \psi t + \frac{1}{2} u \cdot u + \frac{\phi}{\epsilon} + g z \right) = 0 \]

Generalized Bernoulli Theorem:

\[ \psi_t + \frac{1}{2} \nabla \cdot u^2 + \frac{\psi}{\epsilon} + g z = F(t) \quad \text{some constant field only a function of time.} \]

Neglect:\n
\[ \frac{\beta}{z} \ll 1 \]

Evaluate at the free surface:\n
\[ \psi = \psi_{\text{atm}}, \quad z = h \]

\[ \frac{\partial}{\partial t} \psi(x,y,t) = -gy + \left( F - \psi_{\text{atm}} \right) \]

Note: $F$ influences $\psi$ but not $\nabla \phi$ and can be chosen at our convenience $\Rightarrow$ assume $F = \psi_{\text{atm}} / \epsilon$

\[ \psi_t(x,0,t) = -g y \]

DBC - S
Summarizing the math problem

Solve \( \nabla^2 \phi = 0 \) subject to

KBC - b \( \frac{\partial \phi}{\partial z} (x, -H, t) = 0 \)

KBC - s \( \frac{\partial \phi}{\partial t} (x, 0, t) = \frac{\partial \eta}{\partial t} \)

DBC - s \( \frac{\partial \phi}{\partial t} (x, 0, t) = -g \eta \)

Search for wave-like solutions, with \( \eta = \eta_0 \cos(kx - \omega t) \)

\( \Rightarrow \eta_t = \omega \eta_0 \sin(kx - \omega t) \)

Guess a solution of the form

\( \phi = \Phi(z) \sinh(kx - \omega t) \)

Then \( \nabla^2 \phi = 0 \) \( \Rightarrow \) \( \Phi_{zz} - k^2 \Phi = 0 \)

which has general solution

\( \Phi = \Phi_1 e^{kz} + \Phi_2 e^{-kz} \)

\( \uparrow \quad \uparrow \quad \text{constants} \)
Applying KBC - B & KBC - S
after some manipulation you can show

\[ \phi = \Phi_0 \cos h \left[ k (z + H) \right] \sin (kx - \omega t) \]

\[ \text{constant} = \frac{\Phi_0 \omega}{k \sinh (kH)} \]

But the solution is incomplete because we need to ensure that \( k \) and \( \omega \) satisfy DBC - S

Combine DBC - S & KBC - S and eliminate \( \Phi_0 \) to find

\[ \left[ \phi_{xx} = -g \phi_z \right]_{z = 0} \]

Plugging in our solution for \( \phi \) gives

\[ \omega^2 = g k \tanh (kH) \]

In general \( \omega = \omega(k) \) is called the "dispersion relation" (why?)
\[ \tanh(kH) \to kH \]

Shallow water wave:
\[ \omega^2 \approx g k H \]

\[ \Rightarrow C_p = \frac{\omega}{k} = \sqrt{g H} \]

Deep water wave:
\[ \omega^2 \approx g k \]

\[ \Rightarrow C_p = \frac{\omega}{k} = \sqrt{g k} \]
**Deep Water Wave** \( H \gg \lambda \) (Wavelength)

- \( u + w \) same magnitude
- Both decay away from surface as \( \exp(\kappa z) \)
- Parcel paths are circles
- \( C_p = \sqrt{\frac{g}{\kappa}} = \sqrt{\frac{g}{2\pi}} \)

\[ \Rightarrow \text{waves with longer wavelength travel faster; "dispersive"} \]

\[ \eta (z=0) \]

**Shallow Water Wave** \( H \ll \lambda \)

- \( \omega \ll \Omega \)
- \( u \approx \text{const. with depth} \)
- Parcel paths are \( x \) lines

- \( C_p = \sqrt{gH} \)

\( \Rightarrow \) waves of all wavelengths travel same speed

(Faster than deep water waves)

"Non-dispersive"