Velocity, Eulerian & Lagrangian points of view, 
the Material Derivative

Motivation: Conservation laws (like $E = mc^2$) most naturally apply following a fluid parcel [defined as a "Lagrangian" coordinate system]. But measurements and mathematical tools describe changes at fixed points in space [defined as an "Eulerian" coordinate system].

Define the position of a fluid parcel

$L = \text{Lagrangian}$

$\mathbf{x}_L = \mathbf{x}_L\left(\mathbf{x}_0, t\right)$

position at $t = 0$

parcel path

$t = 0$

$\varepsilon_0$ is a way of labeling a parcel
Velocity is the rate of change of position following a fluid parcel

\[
\text{velocity} \equiv \left| \frac{\partial \mathbf{x}}{\partial t} \right| = \left| \mathbf{u} \right|
\]

and if we consider all possible \( \mathbf{x} \), then \( \mathbf{u} \) spans all \( \mathbf{x} \) and \( t \) of the fluid system, and we have the velocity field \( \mathbf{u}(\mathbf{x}, t) \) [m s\(^{-1}\)] vector field

\[
(\mathbf{u}, v, w) = \mathbf{u}(\mathbf{x}, t) + \mathbf{j}w(\mathbf{x}, t) + \mathbf{k}v(\mathbf{x}, t)
\]

Now consider small changes in the value of some scalar field \( T(\mathbf{x}, t) \), using the chain rule

\[
\delta T = \frac{\partial T}{\partial t} \delta t + \frac{\partial T}{\partial x} \delta x + \frac{\partial T}{\partial y} \delta y + \frac{\partial T}{\partial z} \delta z
\]

where \( \delta x \) is an arbitrary change in position.
\[
\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{\delta x}{\delta t} + \frac{\partial T}{\partial y} \frac{\delta y}{\delta t} + \frac{\partial T}{\partial z} \frac{\delta z}{\delta t}
\]

then assume \( \delta x \) is not arbitrary, but is following a fluid parcel, so \( \frac{\delta x}{\delta t} = u \) \( (\frac{\delta x}{\delta t} = u, \text{ etc...}) \)

In this special case we use notation \( \frac{\delta T}{\delta t} = \frac{D T}{D t} \)

and
\[
\frac{D T}{D t} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}
\]
\[
= T_t + u \cdot \nabla T \quad \text{\( \left( \text{Dot product} \ \vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \)}
\]
\[
= T_t + u_i \frac{\partial T}{\partial x_i} \quad \text{\( \left( \text{Indicial notation: sum on repeated indices} \)}
\]

In words \( \frac{D T}{D t} \) is the "rate of change of \( T \) following a fluid parcel."

And \[
\frac{D (\cdot)}{D t} = \left\{ \frac{\partial (\cdot)}{\partial t} + u \cdot \nabla (\cdot) \right\} \quad \text{is the "Material Derivative"}
\]

\( \frac{\partial}{\partial x} \) and etc in \( (*) \) are Eulerian, meaning in coordinates fixed in space.

NOTE: \( \frac{\partial}{\partial x} \) and etc in \( (*) \) are Eulerian, meaning in coordinates fixed in space.