Simpler case: make the parcel a cube of volume $\delta V$

$$-\hat{n} \delta A = \delta V$$

$$-p \hat{n} \delta A = -\hat{i} p_x \delta A$$

where $\hat{i} = (1,0,0)$ unit vector in $x$-direction

Net $x$-force $= -(p_x - p_1) \delta A = -(p_x - p_1) \delta y \delta z = -(p_x - p_1) \frac{\delta V}{\delta x}$

or

$$\frac{x\text{-force}}{\text{unit volume}} = \lim_{\delta V \to 0} \frac{\text{net x-force}}{\delta V} = \lim_{\delta x \to 0} \frac{-(p_x - p_1)}{\delta x} = -\frac{\partial p}{\partial x}$$

(higher pressure on this side ($\partial p/\partial x > 0$) pushes cube to the left)

Same argument for $y$- and $z$-directions gives...

$$\frac{x\text{-force}}{\text{unit volume}} = -\hat{i} \frac{\partial p}{\partial x} - \hat{j} \frac{\partial p}{\partial y} - \hat{k} \frac{\partial p}{\partial z} = -(\frac{\partial p_x}{\partial x}, \frac{\partial p_y}{\partial y}, \frac{\partial p_z}{\partial z})$$

$$= -(p_x, p_y, p_z) \text{ subscript notation}$$

$$= -\nabla p \text{ where } \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \text{ "grad" the gradient operator}$$
Hydrostatic balance: pressure field balances gravity

Parcel has mass:

\[ m = \rho \cdot \delta V \]

\[ \text{z-force from pressure} = -\hat{k}\frac{\partial p}{\partial z} \]

\[ \text{unit volume} \]

\[ \text{z-force from gravity} = -\hat{k} \frac{mg}{\text{unit vol.}} = -\hat{k} \rho g \]

\[ 9.8 \text{ m/s}^2 \]

\[ \text{total z-force} \]

\[ \frac{\text{unit vol.}}{0} = -\hat{k} \frac{\partial p}{\partial z} - \hat{k} \rho g \]

implies no acceleration of the parcel

\[ \frac{\partial p}{\partial z} = -\rho g \]

The "hydrostatic balance"

- simplest momentum equation
- works even when \(\rho\) is not constant
- excellent approximation for 95% of
  Atmos. & Ocean. flow!
Example: pressure field in a tank of water

\[ p = p_{\text{atm}} \text{ in the air} \]

\[ \int_0^z \left[ \frac{dp}{dz'} = -\rho g \right] dz' \]

\[ \Rightarrow p(0) - p(z) = -\rho g (0-z) \]

\[ \Rightarrow p(z) = p_{\text{atm}} - \rho g z \]

Note: \( z \) is negative in the water

Note: \( p_{\text{atm}} \approx 10^5 \text{ Pa} = 1 \text{ bar} = \text{weight of 10,000 kg of air/m}^2 \)!
(at sea level)

and for water \( \rho_o \approx 1000 \text{ kg m}^{-3} \)

\[ \Rightarrow p \to 2 \times p_{\text{atm}} \text{ at 10 m depth} \]