OCTAHEDRAL SHEAR STRESS CRITERION (VON MISES)

Since hydrostatic stress alone does not cause yielding, we can find a material plane called the octahedral plane, where the stress state can be decoupled into dilation strain energy and distortion strain energy\(^1\). On the octahedral plane, the octahedral normal stress solely contributes to the dilation strain energy and is

\[
\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}
\]  

(1)

This is the average of the three principal stresses. For example, if \(\sigma_1 = \sigma_2 = \sigma_3 = p\) where \(p\) is the pressure, then \(\sigma_h = p\). The remaining strain energy in the state of stress is determined by the octahedral shear stress and is given by

\[
\tau_h = \frac{1}{\sqrt{2}} \left( \frac{1}{3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \right)
\]  

(2)

We expect yielding when the octahedral shear stress is equal to or exceeds a stress criterion value for failure for a given material, which is the octahedral stress criterion \(\tau_{h0}\):

\[
\begin{align*}
\tau_h &\geq \tau_{h0} \quad \text{(failure)} \quad \text{(3)} \\
\tau_h &= \tau_{h0} \quad \text{(at yielding)} \quad \text{(4)}
\end{align*}
\]

The octahedral stress criterion for say 1080 Al is not likely to be reported in the literature so we need to relate it to the axial yield strength \(\sigma_0\). For a given material under axial load where \(\sigma_1 = \sigma_0\) and \(\sigma_2 = \sigma_3 = 0\), we assume that yielding occurs when the octahedral shear stress is equivalent to the octahedral stress criterion. This means we can combine Eq. 2 and 4 to get the octahedral stress criterion in terms of the yield strength:

\[
\tau_{h0} = \tau_h = \frac{1}{3} \sqrt{(\sigma_0 - 0)^2 + (0 - 0)^2 + (0 - \sigma_1)^2} = \frac{\sqrt{2}}{3} \sigma_0
\]  

(5)

With \(\sigma_0 = \frac{1}{\sqrt{2}} \tau_{h0}\), we expect to observe yielding in a material under 3-D loading when, as before, we combine Eq. 2 and 4 to get

\[
\sigma_0 = \frac{1}{\sqrt{2}} \sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}
\]  

(6)

As a result, we can define the effective stress for von Mises theory to be equivalent to Eq. 6.

\[
\bar{\sigma}_h = \frac{1}{\sqrt{2}} \sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}
\]  

(7)

\(^1\) For dilation, stresses are the same in all directions and there is no shear. For distortion, stresses are different in magnitude and/or direction and so there exists shear stress. See full derivation in Popov, E.P., 1968 *Introduction to Mechanics of Solids*, 1st edition, Prentice Hall, Englewood Cliffs, NJ.
We can express Eq. 7 in terms of the stress invariants ($I_1$ and $I_2$):

$$\bar{\sigma}_H = \frac{1}{\sqrt{2}} \sqrt{2(l_1 - 3l_2)^2}$$

(8)

Failure is likely when

$$\sigma_H \geq \sigma_0$$

(9)

For plane stress ($\sigma_3 = 0$), we expect yielding when $\sigma_H = \sigma_0$ and so

$$\sigma_0 = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - 0)^2 + (0 - \sigma_1)^2}$$

$$\sigma_0^2 = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 \right]$$

$$\sigma_0^2 = \frac{1}{2} \left[ \sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 + \sigma_1^2 \right]$$

$$\sigma_0^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

(10)

The last form in Eq. 10 is an ellipse with its major axis along the $\sigma_1 = \sigma_2$ line. We can solve this and graph it in Mathematica:

```plaintext
Eqn = Solve[s1^2 - s1*s2 + s2^2 == 1, s2]
{\{s2 -> \frac{1}{2} \left[ s1 - \sqrt{4 - 3s1^2} \right], s2 -> \frac{1}{2} \left[ s1 + \sqrt{4 - 3s1^2} \right]\}}
Eqn[[1]][[1]][[2]]
\frac{1}{2} \left[ s1 - \sqrt{4 - 3s1^2} \right]
Plot[{Eqn[[1]][[1]][[2]], Eqn[[2]][[1]][[2]]}, {s1, -2\sqrt{3}/3, 2\sqrt{3}/3},
AxesLabel -> {s1, s2}, AspectRatio -> 1]
```

Figure 1. Mathematica code to plot octahedral stress failure for plane stress.