These notes contain the material covered in class concerning the matrix exponential and stability of linear systems, with a bit of generality.

To begin, be aware that all systems are nonlinear. Linear models are constructed because, under appropriate conditions, they are reasonable approximations of the true system that have easier tools associated with them. Recall that the form of a general nonlinear system is

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u).
\end{align*}
\]

Questions that we typically would like to answer about systems are:

1. What are the functions \(x(t)\) and \(y(t)\)?
2. How do we simulate the system evolution (numerically)?
3. How do we change the system behavior by choosing \(u(t)\)?

With respect to the second item, it is important to be aware that numerical simulations of the system equations \(x(t)\) and \(y(t)\) based on the differential equations are subject to error if the simulations are not run with the appropriate simulation method (this topic is not the subject of this course, just something to be aware of). In particular, the development of effective numerical simulation techniques for nonlinear systems is an active area of research.

Now, for a linear system, which has the form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

the above three questions have reasonably straightforward answers (compared to nonlinear systems). Only the first question will be addressed for now.

To begin, consider the scalar linear system

\[
\begin{align*}
\dot{x}(t) &= ax(t) + bu(t) \\
y(t) &= cx(t) + du(t).
\end{align*}
\]

If \(u(t) = 0\), we can directly solve the equation for \(x\):

\[
x(t) = e^{at}x(0).
\]

The equation for the output \(y\) is then

\[
y(t) = ce^{at}x(0).
\]

If the control is not zero, this solution generalizes to

\[
\begin{align*}
x(t) &= e^{at}x(0) + e^{at} \int_0^t e^{-a\tau}bu(\tau)d\tau, \\
y(t) &= ce^{at}x(0) + ce^{at} \int_0^t e^{-a\tau}bu(\tau)d\tau + du(t).
\end{align*}
\]
\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \]
\[ y(t) = Cx(t) + Du(t) \]

In order to extend this scalar result to vector systems of the form
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

recall the Taylor series expansion of a scalar function \( f(x(t)) \):
\[ f(x(t)) = f(x_0) + \left( \frac{\partial f(x(t))}{\partial x(t)} \right)_{x(t)=x_0} (x(t) - x_0) + \frac{1}{2!} \left( \frac{\partial^2 f(x(t))}{\partial x(t)^2} \right)_{x(t)=x_0} (x(t) - x_0)^2 + \ldots \]

The equation for a scalar exponential function is then
\[ e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \ldots \]

As it turns out, this formula can be extended directly to linear time-invariant systems with the definition
\[ e^{At} \equiv I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots \]

which is referred to as the matrix exponential. This definition is in fact defines the solution to the equation \( \dot{x} = Ax \) by \( x(t) = e^{At}x(0) \) as we can see by just verifying directly:
\[ \dot{x} = \frac{d}{dt}(e^{At}x(0)) \]
\[ = \frac{d}{dt}(e^{At})x(0) \]
\[ = \frac{d}{dt} \left( I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots \right) x(0) \]
\[ = \left( 0 + A + A^2t + \frac{1}{2}A^3t^2 + \ldots \right) x(0) \]
\[ = A \left( I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots \right) x(0) \]
\[ = Ae^{At}x(0) \]
\[ = Ax. \]

One can similarly verify that if the control is not zero, then the solution for a linear time-invariant system is
\[ x(t) = e^{At}x(0) + e^{At} \int_0^t e^{-A\tau}Bu(\tau)d\tau \]
\[ y(t) = Ce^{At}x(0) + Ce^{At} \int_0^t e^{-A\tau}Bu(\tau)d\tau + Du(t). \]
Note: The integral of a matrix is the matrix of the integrals of the matrix components:

$$\int_0^t M(\tau)\,d\tau = \left[ \begin{array}{ccc} m_{11}(\tau) & \ldots & m_{1n}(\tau) \\ \vdots & & \vdots \\ m_{n1}(\tau) & \ldots & m_{nn}(\tau) \end{array} \right]$$

$$\int_0^t M(\tau)\,d\tau = \left[ \begin{array}{ccc} \int_0^t m_{11}(\tau)\,d\tau & \ldots & \int_0^t m_{1n}(\tau)\,d\tau \\ \vdots & & \vdots \\ \int_0^t m_{n1}(\tau)\,d\tau & \ldots & \int_0^t m_{nn}(\tau)\,d\tau \end{array} \right]$$

As an example, consider the case where we have

$$A = \left[ \begin{array}{ccc} \lambda_1 & 0 & \ldots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{array} \right]$$

For the 2x2 case (as an example) with real eigenvalues (as opposed to complex pairs) we have

$$e^{At} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{array} \right] + \left[ \begin{array}{cc} \lambda_1^2 t^2/2 & 0 \\ 0 & \lambda_2^2 t^2/2 \end{array} \right] + \left[ \begin{array}{cc} \lambda_1^3 t^3/3! & 0 \\ 0 & \lambda_2^3 t^3/3! \end{array} \right] + \ldots$$

$$= \left[ \begin{array}{cc} 1 + \lambda_1 t + \lambda_1^2 t^2/2 + \lambda_1^3 t^3/3! + \ldots & 0 \\ 0 & 1 + \lambda_2 t + \lambda_2^2 t^2/2 + \lambda_2^3 t^3/3! + \ldots \end{array} \right]$$

$$= \left[ \begin{array}{cc} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{array} \right]$$

Clearly for this system, stability is determined by the values $\lambda_i$:

$$\begin{array}{ccc} \lambda_i & < & 0, \text{ asymptotically stable} \\
& = & 0, \text{ stable} \\
& > & 0, \text{ unstable} \end{array}$$

Two other cases are of interest. The first is matrices that have the form

$$A = \left[ \begin{array}{cc} a & b \\ -b & a \end{array} \right].$$

One can verify in this case that the resulting system state with $u = 0$ is given by

$$x(t) = e^{at} \left[ \begin{array}{cc} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{array} \right] x(0).$$

Stability is again determined by the sign of the constant $a$. The final case is

$$A = \left[ \begin{array}{cc} a & 1 \\ 0 & a \end{array} \right].$$

Directly computing the matrix exponential for this system we have

$$x(t) = \left[ \begin{array}{cc} e^{at} & te^{at} \\ 0 & e^{at} \end{array} \right] x(0).$$
Clearly this system will be unstable if \( a > 0 \). If \( a = 0 \), it will also be unstable as the term \( te^{at} \) becomes simply \( t \). If \( a < 0 \), however, the system will be asymptotically stable because the decaying exponential will dominate the growing linear time term. Now we would like to show that we can always reduce our generic systems with arbitrary \( A \) matrix to the cases that we have above.

To begin recall the construction of eigenvectors, \( v \), and eigenvalues, \( \lambda \):

\[
\lambda v = Av \\
(\lambda v - Av) = (\lambda I - A)v = 0.
\]

This equation is solved either if \( v = 0 \) or if \( \lambda I - A = 0 \). The case where \( v = 0 \) is referred to as a degenerate case, and is only one solution to the equality. The second condition holds if \( \det(\lambda I - A) = 0 \). If this is true, then \( v \) can be anything. The equation given by expanding \( \det(\lambda I - A) = 0 \) is referred to as the characteristic equation of the system. The values \( \lambda \) are referred to as the system eigenvalues. For every eigenvalue, there is a corresponding unique eigenvector.

**Note:** eigenvalues are in fact complex: \( \lambda_i = a_i + jb_i \), \( j = \sqrt{-1} \) and complex numbers occur in conjugate pairs (and therefore eigenvalues do as well).

Example: (pendulum or RL circuit) Let \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). The characteristic equation of the system is

\[
\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0
\]

The solutions to this equation are \( \lambda = \pm j \). These eigenvalues have no real part. Note that we have computed the solution to this system earlier and it is given by strictly sine and cosine terms (no exponentials).

The eigenvectors are found from the equation \( \lambda x = Ax \):

\[
j \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Now, to calculate the eigenvector, we can write

\[
jx_1 = x_2 \\
jx_2 = -x_1
\]

To reach this solution, we chose a value for \( x_1 (x_1 = 1) \) and then determined the appropriate value for \( x_2 \). To show that this worked, just check: \( jx_2 = j \cdot j = -1 = -x_1 \). The other eigenvector for the system is \( x = \begin{bmatrix} -j \\ 1 \end{bmatrix} \).

The characteristic equation for any system can always be decomposed into the form

\[
\det(\lambda I - A) = (s + \lambda_1)^{n_1} + (s + \lambda_2)^{n_2} + \ldots + (s + \lambda_k)^{n_k}(s^2 + a_{nk+1}s + b_{nk+1})^{n_{k+1}} \ldots (s^2 + \ldots)^{n_p}
\]

which is a product of terms that correspond to strictly real eigenvalues which occur \( n_i \) times and to complex pairs that occur \( n_p - n_k \) times.

Let \( v_i \) be the eigenvector that corresponds to eigenvalue \( \lambda_i \). Note that we can have \( \lambda_1 = \lambda_2 \) but \( v_1 \neq v_2 \).

Now recall \( v\lambda = Av \) so we can write

\[
\begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix} = A \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}
\]
To see where we got this representation, note that $v_1\lambda_1 = Av_1$ and $v_2\lambda_2 = Av_2$. Then

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = A \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{21} \\ \lambda_1 v_{12} & \lambda_2 v_{22} \end{bmatrix} = A[v_1 \ v_2]$$

We can then write

$$S = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = V^{-1}AV$$

or $A = VSV^{-1}$.

Now when we consider the matrix exponential of a generic $A$ matrix we can write

$$e^{At} = e^{VSV^{-1}t} = Ve^{St}V^{-1}$$

where $S$ is diagonal and we can use the earlier discussed methods using it rather than the original $A$. To see the final step (pulling out the matrices $V$ and $V^{-1}$ from the matrix exponential) note the following from the definition of the matrix exponential:

$$e^{VSV^{-1}t} = I + (VSV^{-1})t + (VSV^{-1})^2t^2/2 + (VSV^{-1})^3t^3/3! + \ldots$$

$$= I + (VSV^{-1})t + (VSV^{-1}VSV^{-1})t^2/2 + (VSV^{-1}VSV^{-1}VSV^{-1})t^3/3! + \ldots$$

$$= I + (VSV^{-1})t + (VSSV^{-1})t^2/2 + (VSSSV^{-1})t^3/3! + \ldots$$

$$= V(I + St + S^2t^2/2 + \ldots)V^{-1}$$

$$= Ve^{St}V^{-1}.$$  \(1\)

Computing the eigenvalues for a generic $A$ matrix gives

$$\det(\lambda I - A) = \det(\lambda VIV^{-1} - VSV^{-1})$$

$$= \det(V(\lambda I - S)V^{-1})$$

$$= \det(V) \det(\lambda I - S) \det(V)^{-1}$$

$$= \det(\lambda I - S)$$

where the last step follows because, for a constant matrix, $\det(V^{-1}) = \frac{1}{\det(V)}$. Clearly, eigenvalues do not change if we change coordinates with a linear transformation.

Consider then the three stability cases discussed earlier. Let’s start with the damped oscillator: $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. The eigenvalues for this system come from

$$\det(sI - A) = \det \begin{bmatrix} s-a & -b \\ b & s-a \end{bmatrix}$$

$$= s^2 - 2as + (a^2 + b^2) = 0.$$
Solving this equation gives \( s = a \pm bj \) so

\[
S = \begin{bmatrix} a + bj & 0 \\ 0 & a - bj \end{bmatrix}.
\]

Now note from Eq. (1), that the effect of the transform between the original and diagonal system is just pre and post multiplication by constant matrices. Clearly these multiplications will not affect stability. Therefore the stability of systems with complex eigenvalues is determined by the sign of the real part of the eigenvalues.

For the next case, consider the situation with real eigenvalues on the diagonal: \( A = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \).

This system is already in diagonal form with the eigenvalues on the diagonal and has stability determined by whether the eigenvalues (which are strictly real), are positive, zero or negative.

Finally, consider the case \( A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \). One can check that the eigenvalues for this system are both \( \lambda = a \). So systems with repeated real eigenvalues will be unstable unless the eigenvalues are strictly negative. In particular, repeated eigenvalues at the origin will correspond to unstable systems.

To summarize then, stability of systems is determined by eigenvalues, specifically the real part of the eigenvalues, as follows:

\[
Re(\lambda_i) < 0, \text{ for all } i, \text{ asymptotically stable} \\
\leq 0, \text{ for all } i, \text{ no repeated eigenvalues with zero real part, stable} \\
\leq 0, \text{ for all } i, \text{ but repeated eigenvalues with zero real part, unstable} \\
> 0, \text{ for any } i, \text{ unstable}
\]