Problem 1

(a) The state equations for this system are:
\[
\frac{d}{dt}x = \frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{l+R}\dot{\theta} \\ -g\sin(\theta) - R\dot{\theta}^2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}
\]

(b) Linearizing about the point \( x^0 = [\theta^0, \dot{\theta}^0]^T = [0, 0]^T \) gives the following:
\[
A = \left[ \frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \frac{\partial f_2}{\partial x_2} \right]_{x^0} = \left[ \frac{\partial f_1}{\partial \theta} \frac{\partial f_1}{\partial \dot{\theta}} \frac{\partial f_2}{\partial \theta} \frac{\partial f_2}{\partial \dot{\theta}} \right]_{\theta^0, \dot{\theta}^0} = \left[ \begin{array}{cc} 0 & 1 \\ \frac{1}{l+R}(-g\cos(\theta)) & \frac{1}{l+R}(-2\dot{\theta}) \end{array} \right]_{0,0}
\]
\[
A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}
\]
which gives the desired result \( \ddot{\theta} + \frac{g}{l}\theta = 0 \).

(c) To show this result by using the small value approximation, note that for small values of \( \theta \), \( \sin(\theta) \approx 1 \) and \( l + R\theta \approx l \). Similarly \( \dot{\theta}^2 \approx 0 \). So we have
\[
(l + R\theta)\ddot{\theta} + g\sin(\theta) + R\dot{\theta}^2 \approx \ddot{\theta} + \frac{g}{l}\theta = 0
\]

Problem 2

(a) We solve for the equilibrium state by setting the three functions to zero:
\[
\begin{align*}
\dot{x}_1 &= f_1(x) = G(u_1 - x_1) + u_2 - x_3 = 0 \\
\dot{x}_2 &= f_2(x) = x_3 = 0 \\
\dot{x}_3 &= f_3(x) = x_1 - x_2 - r(x_3) = 0
\end{align*}
\]
From the second equation we immediately see that \( x_3^0 = 0 \). Substituting this value and the given values of \( u_1^0 = 1 \), \( u_2^0 = 27 \) we have
\[
\begin{align*}
G(1 - x_1) + 27 &= 0 \\
x_1 - x_2 - r(0) &= 0
\end{align*}
\]
From the given function for \( G \), we have \( G(1 - x_1) = (1 - x_1)^3 \) and from the plot for \( r \) we have \( r(0) = 2 \). Thus
\[
\begin{align*}
(1 - x_1)^3 + 27 &= 0 \\
x_1 - x_2 - 2 &= 0
\end{align*}
\]
Solving the first gives \( x_1^0 = 4 \) and then solving the second gives \( x_2^0 = 2 \).
(b) The task here is to linearize the system about the equilibrium point that we just found. We are thus looking for the two matrices

\[
\begin{align*}
A &= \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}
\end{bmatrix} \bigg|_{x^0,u^0},
B &= \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\
\frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2}
\end{bmatrix} \bigg|_{x^0,u^0}
\end{align*}
\]

For \( f_1 \) we must substitute in the function for \( G \):

\[
f_1(x) = (u_1 - x_1)^3 + u_2 - x_3
\]

Evaluating the terms for the first rows of \( A \) and \( B \) gives:

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= 3(u_1 - x_1)^2(-1) = -3(u_1 - x_1)^2 \\
\frac{\partial f_1}{\partial x_2} &= 0 \\
\frac{\partial f_1}{\partial x_3} &= -1 \\
\frac{\partial f_1}{\partial u_1} &= 3(u_1 - x_1)^2 \\
\frac{\partial f_1}{\partial u_2} &= 1
\end{align*}
\]

For the second rows with \( f_2 = x_3 \) we have

\[
\begin{align*}
\frac{\partial f_2}{\partial x_1} &= 0 \\
\frac{\partial f_2}{\partial x_2} &= 0 \\
\frac{\partial f_2}{\partial x_3} &= 1 \\
\frac{\partial f_2}{\partial u_1} &= 0 \\
\frac{\partial f_2}{\partial u_2} &= 0
\end{align*}
\]

And for the last rows with \( f_3 = x_1 - x_2 - r(x_3) \) we have

\[
\begin{align*}
\frac{\partial f_3}{\partial x_1} &= 1 \\
\frac{\partial f_3}{\partial x_2} &= -1 \\
\frac{\partial f_3}{\partial x_3} &= -\frac{\partial r(x_3)}{\partial x_3} = -1 \\
\frac{\partial f_3}{\partial u_1} &= 0 \\
\frac{\partial f_3}{\partial u_2} &= 0
\end{align*}
\]

Now we must evaluate each of these at the equilibrium point, and for the third element in this last set we must use the chart. Note from the chart that this partial derivative corresponds to the slope of the plot in the area around the equilibrium point: 1. We thus have the linear system matrices

\[
\begin{align*}
A &= \begin{bmatrix}
-27 & 0 & -1 \\
0 & 0 & 1 \\
1 & -1 & -1
\end{bmatrix},
B &= \begin{bmatrix}
27 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\end{align*}
\]
**Problem 3**

Using Mathematica, the linearization about the hover equilibrium is:

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-D}{m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix}
\]

**Problem 4**

The inverted pendulum and cart system is one that is used often to demonstrate the coupling of control and dynamics. In this problem, the concepts of linearization and simulation are examined, noting the tools (namely, MATLAB) that are available to study nonlinear systems.

(a) The `linmod` command in MATLAB allows for numerical linearization of nonlinear systems about an equilibrium point. The output is a linear state-space model of a system. For more references on how to use `linmod`, see the Mathworks web site (www.mathworks.com) and search for “linmod.”

The system dynamics can be expressed in Simulink, as done in `hw4cartpend.mdl`, shown below in Fig. 1: The `linmod` command in MATLAB using the syntax:

![Figure 1: The full nonlinear dynamics of the inverted pendulum and cart problem.](image)

\[
x_0 = [0, \pi, 0, 0]; u_0 = 0; [A, nl, B, C, D] = \text{linmod}('hw4cartpend', x_0, u_0);
\]

where the subscript “nl” represents “nonlinear,” generates the state-space representation of the nonlinear system. Note that the linearization occurs about the equilibrium point, \( x_0 = [0, \pi, 0, 0] \), which corresponds to the pendulum standing straight up. The resulting process matrix, \( A_{nl} \), is given by:

\[
A_{nl} = \begin{bmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 2.6755 & -0.1818 & 0 \\ 0 & 31.2136 & -0.4545 & 0 \end{bmatrix}
\]
Using the provided linearized model (given in the homework statement), and evaluating for the given numerical values (also given), we find our linearized process matrix, $A$, to be identical to the one given by MATLAB for $A_{nl}$ (which is a good thing!).

Hence, we find that the eigenvalues (using `eig` command) for both the analytical and numerical linearization of the system dynamics are:

$$\text{eigenvalues } = [0, 5.5680, -5.6069, -0.1428]$$

The thing to note here is that the system contains a positive eigenvalue ($\lambda = 5.5680$), which corresponds to instability of the pendulum “up” position.

**(b)** As noted above, the pendulum “up” position is unstable as witnessed by the presence of the positive eigenvalue, and by the open-loop response in Fig. 2. We can see immediately that the position of the cart, $x$, increases without bound. We can also examine the Bode diagram (Figure 2) for the open-loop system to see that we can have instabilities. Note in the magnitude plot that we have positive gain for the low frequencies, which means the input error signal is amplified, rather than attenuated. Obviously, this is not desired, and we would like to be able to apply feedback control so as to produce a stable system.

By employing a negative feedback control law of the form: $u = -Kx$, we can see that the system becomes:

$$\dot{x} = Ax + Bu = Ax + B(-Kx) = (A - BK)x = \tilde{A}x$$

Now, we can examine the closed-loop stability by finding the eigenvalues of the closed-loop process matrix, $\tilde{A}$, which are given by:

$$\text{Closed-loop eigenvalues } = [-6.6373, -4.5459, -0.9084 \pm 0.8079j]$$

We see that we no longer have eigenvalues with positive real parts, which implies that we now can attain stability using our negative feedback control. We can see this in Fig. 3.

**(c)** Now, we want to examine the effect of different initial conditions on the behavior of the dynamics. This is one of the strengths of simulation, in that we are able to test different setups without spending the time or expense of running actual hardware.
Figure 3: The closed-loop response of the inverted pendulum and cart system.

The simulation model built in Simulink to examine the closed-loop response is shown in Fig. 4. The initial conditions can be varied by changing the “Initial Condition” parameters in each of the “Integrator” blocks within the dynamics model. In other words, there are four integrators (one going from acceleration to velocity, and another from velocity to position, times two for each coordinate) in which the initial conditions can be specified.
Figure 4: Simulink model for the closed loop cart and pendulum system.