This homework set covers stability and performance through a series of application examples. The first problem continues the ducted fan design example by finding and classifying the system equilibria. The second problem provides a set of three real-world models in which you must identify the equilibrium points and determine stability of the equilibrium points (through simulation). The third problem explores performance specification in the context of the cruise control example, including step response and frequency response. The final two problems are to review material from solving second order differential equations.

1. [5 pts] In part (1) of the ducted fan design problem, find the equilibria for the system. Using the simulink model (on the web page), check the stability of these equilibria by choosing initial conditions near (but not equal to) the equilibria and running the simulation. There will, in fact, be infinitely many equilibria, but they will all fall into one of two categories. Turn in representative plots of determining the stability type of each of the two classes (one plot for each class).

2. [15 pts] For each of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable, or unstable. To determine stability, you can either use a phase portrait (if appropriate) or simulate the system by perturbing the initial condition slightly from the equilibrium point and then seeing how the state evolves. (Note: if you know how to check stability through linearization, you can also use this approach.)
   
   (a) **Duffing equation.** The Duffing equation is a model for a nonlinear mass spring system:
   \[ m\ddot{x} = k(x + ax^3) - cx, \]
   where \( m = 1000 \) kg is the mass, \( k = 250 \) N/sec\(^2\) is the nominal spring constant, \( a = 10 \) represents the nonlinearity of the spring, and \( c = 1 \) N/sec is the damping coefficient. Note that this is very similar to the mass spring systems we have studied in class, except for the nonlinearity.
   
   (b) **Modified Predator-Prey ODE.** In class we saw an ODE model for the predator-prey problem:
   \[
   \begin{align*}
   \dot{x}_1 &= b_r x_1 - ax_1 x_2 - bx_1^2 \\
   \dot{x}_2 &= ax_1 x_2 - d_f x_2 - bx_2^2
   \end{align*}
   \]
   Use the following parameters: \( b_r = 0.7, \ d_f = 0.5, \ a = 0.007, \ b = 0.0005. \)
   
   (c) **Pendulum.** The equations of motion for a single inverted pendulum are given by
   \[ ml^2 \ddot{\theta} = -b\dot{\theta} - mgl\sin(\theta) \]
   where \( \theta \) is the angle of the pendulum (\( \theta = 0 \) rad corresponds to pointing down), \( m = 1 \) kg is the mass of the pendulum (assumed concentrated at the end), \( l = 0.5 \) m is the length of the pendulum, \( b = 0.25 \) N-m-sec\(^{-1}\) is the damping coefficient, and \( g = 9.8 \) m/sec\(^2\) is the gravitational constant.

3. (MATLAB/SIMULINK) You should use `hw1cruise.mdl` to solve this problem, available on the course web page. Consider the cruise control system from Homework Set #1. Set the gains of the system to the values \( K_i = 100, \ K_p = 500. \)
   
   (a) [5 pts] Plot the step response of the system (from 55 m/s to 65 m/s) and measure the rise time, overshoot, settling time, and steady state error.
(b) [5 pts] Modify the block diagram to allow a sinusoidal reference signal superimposed on top of a commanded reference (so that you get something that oscillates around the nominal speed of 55 m/s). Plot the response of the system to a commanded reference speed that varies sinusoidally between 50 m/s and 60 m/s at a frequency of 1 Hz (about 6 rad/sec). Measure the relative amplitude and phase of the velocity with respect to the commanded input. Your answer should be the ratio of the output amplitude to the input amplitude (after subtracting off the means) and the number of radians of phase “lead” or “lag” between the sinusoids.

(c) [5 pts] Plot the frequency response for the cruise control system, showing the gain (relative amplitude) and phase at the following frequencies (all in rad/sec): 0.01 0.03 0.07 0.1 0.3 0.7 1 3 7 10. Your answer should be in the form of two plots: the relative amplitude (gain) versus frequency and the relative phase versus frequency. Use a logarithmic scale for the frequency and amplitude, and a linear scale for the phase. This set of plots is referred to as a Bode plot.

4. Consider a second order system of the form

$$\ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = u(t)$$

with initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$.

(a) [5 pts] Compute the homogenous solution to this equation ($u(t) = 0$) with initial condition $y_0 = 1$, $\dot{y}_0 = 0$. This is the “impulse response” for this system. Plot the impulse response as a function of time for $\omega_n = 1$, $\zeta = 0.5$.

(b) [5 pts] Compute the response of the system to a sinusoidal input $u(t) = A \sin(\omega t)$. Your result should be analytical (a formula, like the ones I gave in lecture) and you should make sure to keep the effects of the initial conditions.

(c) [5 pts] Now assuming that the initial conditions have died out (i.e., ignoring the homogeneous part of the solution), plot the frequency response of the system on a Bode plot, labelling all relevant points. Note: you can find this solution worked out in many textbooks. You are encouraged to look for the solution, but make sure that you provide a derivation of your results and that you understand them. (Pretend that this might be the type of thing you were asked on a closed book section of the midterm.)

(d) [5 pts] Suppose that we now implement a feedback control law of the form

$$u(t) = k_1 (y - v(t)) + k_2 \dot{y},$$

which is intended to allow us to track a new input $v(t)$ (just like the cruise control example). Rewrite the dynamics of the system with this input in the form

$$\ddot{y} + 2\zeta' \omega'_n \dot{y} + \omega'_n^2 y = v(t)$$

(e) [5 pts] Show that we can set the closed loop natural frequency $\omega'_n$ and damping ratio $\zeta'$ to arbitrary values by adjusting the gains $k_1$ and $k_2$ by giving formulas for the gains in terms of the desired $\omega'_n$ and $\zeta'$.

5. Continuing HW2 problem 4:

(a) [5 pts] Note that the equations in terms of $z$ can be written in diagonalized form so that the $z_1$ terms and $z_2$ terms do not depend on each other and can be solved separately. Solve these linear ODEs for $z_1(t)$ and $z_2(t)$ given initial conditions $z_1(0)$, $\dot{z}_1(0)$, $z_2(0)$, $\dot{z}_2(0)$, and input $u(t) = \sin(\omega t)$.

(b) [5 pts] Plot the amplitude and relative phase of the motion of the first mass as a function of the frequency of the sinusoidal input. You can either solve this problem numerically (by building a simulation and measuring the results) or analytically (using the solution from part 5a and
computing the motion of the first mass). If you use a numerical solution, you can use $m = 250$, $k = 50$, $b = 10$.

Hint: The solution will be a combination of decaying exponentials and sines and cosines. We are not interested in the transient behavior here, so ignore the exponentials. Then recall that the amplitude is the square root of the sums of the amplitude of the sine and cosine term, and the phase is the arctan of the amplitude of the cosine over that of the sine.