Consider the motion of a small model aircraft powered by a vectored thrust engine, as shown below.

Let \([x, y, \theta]\) denote the position and orientation of the center of mass of the fan. We assume that the forces acting on the fan consist of a force \(f_1\) perpendicular to the axis of the fan acting at a distance \(r\) and a force \(f_2\) parallel to the axis of the fan. Let \(m\) be the mass of the fan, \(J\) the moment of inertia, \(\gamma\) the gravitational constant, and \(D\) the damping coefficient. Then the equations of motion for the fan are given by:

\[
\begin{align*}
mx' &= f_1 \cos(\theta) - f_2 \sin(\theta) - D\dot{x} \\
m\dot{y} &= f_1 \sin(\theta) + f_2 \cos(\theta) - m\gamma - D\dot{y} \\
J\dot{\theta} &= rf_1
\end{align*}
\]

It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let \(u_1 = f_1\) and \(u_2 = f_2 - m\gamma\), then the equations become

\[
\begin{align*}
m\dot{x} &= -m\gamma \sin(\theta) - D\dot{x} + u_1 \cos(\theta) - u_2 \sin(\theta) \\
m\dot{y} &= m\gamma(\cos(\theta) - 1) - D\dot{y} + u_1 \sin(\theta) + u_2 \cos(\theta) \\
J\dot{\theta} &= ru_1.
\end{align*}
\]

These equations are referred to as the planar ducted fan equations.

1. Write the equations in first order state space form and find the equilibria.

2. Linearize the equations of motion about the hover position \((\theta = 0)\) with the two inputs and two outputs defined by

\[
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}
\]

3. Write the transfer function matrix \(H_{zu}(s)\) between the inputs and the outputs.

For the remainder of the items, consider only the transfer function \(P(s) := H_{z_1 u_1}\) between \(u_1\) and the \(x\) position of the planar ducted fan. Use the following values for the parameters of the system:

\[
\gamma = 0.52 \text{ m/sec}^2, \quad m = 4.25 \text{ kg}, \quad r = 26 \text{ cm}, \quad J = 0.0475 \text{ kg m}^2, \quad D = 0.5 \text{ kg/sec}
\]

These are roughly the values for the planar ducted fan at Caltech. The reason that gravity \(\gamma\) is not 9.8 \text{ m/sec}^2 is because of the presence of a counterweight to offset the weight of the fan.
4. Determine the poles and zeros for \( P(s) \) and plot them in a pole-zero plot. Show that the open loop system is unstable with respect to initial conditions.

5. Design a controller \( U(s) = K(s) \) or \( u(t) = k(z(t)) \) to stabilize \( P(s) \) with respect to initial conditions.

6. Plot the Bode plot of \( P(s) \). Is the closed loop system stable with unity gain feedback?

7. Design a controller which stabilizes the system about \( \theta = 0 \) and tracks a command signal for the height \( y \). You can construct this controller using either state space methods or frequency based methods.

8. Plot the step response for the system and compute the settling time and percent overshoot for your controller.

9. Plot the Bode plot of the closed loop transfer function for your controller and compute the gain and phase margin.

10. For the compensator that you designed above, plot the Nyquist plot of the open loop transfer function and verify the gain and phase margin that you computed.