The exam consists of four questions, worth a total of 70 points. The point values for each section are shown on the left. The time limit is 1 hour 50 minutes. Budget your time to complete as much of the exam as possible. Please show all of your work, not just a final answer. If time does not permit a complete answer, indicate how you would proceed as explicitly as possible.

The exam is open book. You may use the textbook (Dorf and Bishop), course handouts, lecture and class notes, course problem sets and solutions, and handwritten notes. No other books or materials are allowed.
Problem 1

A system has a transfer function

\[
Y(s) = \frac{s + a}{R(s)} = \frac{s + a}{s^3 + 7s^2 + 15s + 9}
\]

(a) (5 pts) Express this system in control canonical form.

\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -15 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, y = [a, 1, 0] x
\]

(b) (5 pts) Is the system stable? Hint: one root is at s=-3.

The poles of the system are located at \( s = -3, -3, -1 \). All poles are in the left half plane, so the system is stable (even though one is repeated).

(c) (5 pts) When is this system uncontrollable?

Based on the control canonical form, the system is always controllable. However, it is possible for the system to have a lower order if \( a = -3 \) or \( a = -1 \) in which case the zero and one of the poles would cancel.

Problem 2

Consider the block diagram for the drivetrain of a 1976 GMC Astro truck:
Compute the following in terms of $A, B, C, D, E$ and $F$:

(a) (5 pts) The transfer function from $T_l$ to $\dot{\theta}_l$

$$\frac{\dot{\theta}_l}{T_l} = -\frac{BCDE}{1 + DE + EF + ABCDE}$$

(b) (5 pts) The transfer function from $T_e$ to $N$ (Hint: it is not $\frac{1}{s}$)

$$\frac{N}{T_e} = \frac{B(1 + DE + EF)}{1 + DE + EF + ABCDE}$$

**Problem 3**

The transfer function, $G(s)$, of a plant to be controlled is

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^2 + 10s + 5}$$

(a) (5 pts) Write the system in control canonical form.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) (5 pts) Rewrite the system equations with the control input $u = -Kx + l_r r$

where $r$ is a reference command input.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 - k_1 & -10 - k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ l_r \end{bmatrix} r.$$

(c) (10 pts) Using the state variable feedback in the previous part choose the constants so that the steady-state error for a step input is zero, the system has an overshoot of approximately 0.5% and the settling time (2% criterion) is 0.1 second.

The transfer function for this closed loop system is

$$T(s) = \frac{k_3}{s^2 + (k_2 + 10)s + 5 + k_1}$$

The performance specifications are an overshoot of approximately 0.5% and a settling time less than 0.1 second. For the overshoot we have

$$0.005 = \frac{e^{-\zeta \pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \Rightarrow \zeta^2 = \frac{1}{1 + \frac{\pi^2}{(0.005)^2}} = 0.74$$
Therefore $\zeta = 0.86$. The criterion for the 2\% settling time gives
\[
T(s) = \frac{4}{\zeta \omega_n} \leq 0.1 \Rightarrow \omega_n \geq \frac{4}{0.086} = 46.5
\]
The desired characteristic polynomial is then
\[
s^2 + 2(0.86)(46.5)s + (46.5)^2 = s^2 + 80s + 2162.25
\]
Equating the coefficients from $T(s)$ to this equation gives $k_1 = 2157.25$ and $k_2 = 70$. To meet the requirement for zero steady state error, note that the closed loop transfer function is
\[
\frac{Y(s)}{R(s)} = \frac{l_r}{s^2 + 80s + 2162.25}
\]
so we have
\[
e_{ss} = \lim_{s \to 0} s(1-T(s))R(S) = \lim_{s \to 0} s \left( 1 - \frac{l_r}{s^2 + 80s + 2162.25} \right) = 1 - \frac{l_r}{2162.25} = 0 \Rightarrow k_3 = 2162.25
\]
(d) (5 pts) Draw a block diagram of your controller.

[Block diagram image]

**Problem 4**

The asymptotic log-magnitude curves for a transfer function is given in Fig. 1.

(a) (10 pts) Sketch the corresponding asymptotic phase shift curve for the system. Assume that the system has a minimum phase transfer function. (Blank plot is attached).

(b) (10 pts) Determine the transfer function corresponding to the diagram.

A transfer function for part (a), which is not unique, is
\[
G(s) = \frac{140(s+1)}{(s+0.5)(s+7)^2}
\]
Figure 1: Log-magnitude plot for problem 4.

(c) (5pts) Does the system appear to be closed-loop stable if unity gain feedback is applied? (Obviously you will not have an accurate answer, but estimate as best as you can from your plot.)

Yes, there looks to be at least 65 degrees of phase margin, so the system should be stable.