Problem 1. A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming the interest is compounded continuously and the borrower makes payments continuously at a constant annual rate $k$.

(a) Determine the payment rate $k$ that is required to pay off the loan interest in 3 years. (3 points)

(b) Determine how much interest is paid during the 3-year period. (2 pts)

Solution.
(a) Let $S(t)$ be the amount due after $t$ years. Then $S(t)$ satisfies the following DE

$$\frac{dS(t)}{dt} = 0.1 \cdot S(t) - k, \quad S(0) = 8,000. \quad (1)$$

Solving the DE by either using separation of variables or the formula seen in class,

$$S(t) = 8,000e^{0.1t} - (k/0.1)(e^{0.1t} - 1). \quad (2)$$

Hence

$$k = \frac{8,000 \cdot 0.1}{e^{0.3} - 1} \approx 3,086.64$$

(b) The college graduate pays in total $3k = $9,259.92 after three years, hence it paid

$$9,259.92 - 8,000 = \boxed{$1,259.92}$$
in interest.
Problem 2. Assume that the same college graduate in Problem 1 (that borrows $8000 to buy a car to the lender that charges interest at an annual rate of 10%), decides to make payments continuously at a variable annual rate $k(t) = 2t\textsuperscript{1}$. Assume the interest is compounded continuously.

(a) If $S(t)$ denotes the amount owe by the student after $t$ years, find $S(t)$. (4 points)

(b) Determine how much interest is paid during a 2-year period. (1 pt)

Solution.
We have that $S(t)$ satisfies the DE similar to (1), but in this case $k = 2t$ depends on the time $t$. I will present the proof with a more general

$$k(t) = 2at \quad , a > 0.$$ Notice the case $a = 1$ is the one in the quiz. Consider the DE

$$\frac{dS(t)}{dt} = 0.1 \cdot S(t) - 2at, \quad S(0) = 8,000.$$ This is a linear DE, written in standard form,

$$\frac{dS(t)}{dt} - 0.1 \cdot S(t) = -2at, \quad S(0) = 8,000.$$ An integral factor is given by

$$\mu(t) = e^{\int_0^t -0.1ds} = e^{-0.1t}.$$ Hence

$$S(t) = \frac{1}{e^{-0.1t}} \left( \int_0^t e^{-0.1s} \cdot (-2as) ds + 8,000 \right)$$

$$= -2ae^{0.1t} \int_0^t e^{-0.1s} ds + 8,000e^{0.1t}$$

From integration by parts it follows

$$\int se^{-rs} ds = \frac{(rs + 1)}{r^2} e^{-rs} + C, \quad \text{for any} \quad r > 0.$$

\textsuperscript{1}I meant to write $k(t) = 2000t$, what would have made more sense in the problem, but instead I wrote $k(t) = 2t$ in the board. The problem can also be solved for $k(t) = 2t$, but the final number of paid interest do not make much sense.
Taking $r = 0.1$ in the previous formula we get,

$$\int se^{-0.1s} ds = -\frac{(0.1s + 1)}{(0.1)^2}e^{-0.1s} + C, \quad \text{for any } r > 0.$$ 

Then

$$S(t) = 8,000e^{0.1t} + 2ae^{0.1t}\left(\frac{(0, 1t + 1)}{(0, 1)^2}e^{-0.1t} - \frac{1}{(0, 1)^2}\right)$$

$$= 8,000e^{0.1t} - 200ae^{0.1t} + 20at + 200a.$$ 

Finally, we obtain

$$S(t) = (8,000 - 200a)e^{0.1t} + 20at + 200a \quad (3)$$

(b) We find first the deposit function $D(t)$ given by

$$D(t) = \int_0^t 2asds = at^2.$$ 

Hence, after 2 years the college graduate has paid in interest

$$D(2) + S(2) - $8,000 = $1,771.22 - 0.28a.$$ 

For instance, when $a = 1$ he has paid

$$\text{\$1,770.94}$$

in interest.