Problem 1 (5 points). Consider the following DE

\[ y' = y^2(y + 3)(1 - y). \]

(a) Determine all equilibrium solutions and classify them between stable, unstable or neither. (3 points)

(b) Using the classification of the equilibrium solutions draw the direction field of the DE. The direction field must contain all the equilibrium solutions. (2 point)

Solution.

(a) By setting \( y' = 0 \) we find that the equilibrium solutions are

\[ y = -3, \; y = 0 \text{ and } y = 1. \]

To decide whether the equilibrium solution is stable, unstable or neither we analyze the behavior of the derivative of the solutions near the equilibrium solution. Near \( y = -3 \) the derivative changes (as \( y \) increases) from negative to positive values. This implies all solutions are moving away from \( y = -3 \) and hence \( y = -3 \) is an unstable solutions. Near \( y = 0 \) the derivative of all solutions are positive, hence \( y = 0 \) is neither stable nor unstable. Finally, near \( y = 1 \) the derivative changes from positive to negative, hence \( y = 1 \) is an stable solution.

(b) The direction field of the DE is given in Figure 1.

![Figure 1: Direction Field of \( y' = y^2(y + 3)(1 - y) \). The equilibrium solutions \( y = -3, y = 0 \) and \( y = 1 \) are shown in red color.](image-url)
Problem 2 (5 points). Your swimming pool, containing 60,000 gal of water, has been contaminated by 5 kg of nontoxic dye that leaves at swimmer’s skin an unattractive green. The pool’s filtering system can take the water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let \( q(t) \) be the amount of dye in the pool at any time \( t \). (4 points)

(b) How much of the dye is present in the pool after a long time (i.e., when \( t \) goes to infinity)? (1 point) Note: You do not need to solve the DE.

Solution.

(a) The rate at which \( q(t) \) is changing with respect to time is given by

\[
\frac{dq}{dt} = -\frac{q(t)}{60,000} \text{ kg/gal} \times 200 \text{ gal/min} = -\frac{1}{300} \text{ kg/min}. \tag{1}
\]

Notice that the minus sign in Eq. (1), corresponds to the fact that the dye is being removed from the pool. Since initially there are 5 kg of dye on the pool then \( q(0) = 5 \). Finally the initial value problem is given by

\[
\frac{dq}{dt} = -\frac{1}{300}, \quad q(0) = 5.
\]

(b) Notice that \( q = 0 \) is the unique stable equilibrium solution, hence all solutions will tend to 0 as times goes to infinity. This means that there will be no dye present in the pool after a long time.