Sequences

October 1, 2011
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- Sequences do not necessarily start with $a_1$. They may start with any other number.
Basics

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*A sequence is a function* \( f : \mathbb{N} \rightarrow A. \)

- A common notation for a sequence is \( a_1, a_2, \ldots a_n, \ldots \).
- \( a_n \) is usually called the general term.
- Sequences do not necessarily start with \( a_1 \). They may start with any other number.
- A sequence may be finite or infinite.
Describing sequences

There are three common ways to describe sequences:

Explicitly:

1, 3, 5, 7, ..., \(2^n - 1\), ...

\(a_n = 2^n - 1\)

Can you suggest an explicit expression for the general term \(a_n\)?

By a “rule”:

1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, ...

\(a_n = ?\)

Answer: \(a_n\) is “The \(n\)th non perfect square.”

\(a_n\) is the number of different ways to write \(n\) as a sum of no more than \(\lfloor \sqrt{n} \rfloor\) positive integers.
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A sequence:

\[ a_n = n^2 + 2n + 1 \]

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\[a_n = \left\lfloor \sqrt{n} \right\rfloor^2.\]

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By a "rule":

1, 1, 4, 1, 9, 1, 16, ...

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2. Answer: \( a_n \) is “The \( n^{th} \) non perfect square.”
3. \( a_n \) is the number of different ways to write \( n \) as a sum of no more than \( \lfloor \sqrt{n} \rfloor \) positive integers.
Recursively:

- $1, 2, 6, 24, 120, \ldots \quad a_1 = 0, a_n = n a_{n-1}$. 

Remark

The process of constructing a sequence from a given collection $C$, that is building a bijection between $\mathbb{Z}^+$ and $C$ is called enumeration or sequencing.

Example

$(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), (2,1), (3,0), \ldots$ is an enumeration of $\mathbb{N} \times \mathbb{N}$. 

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1. $1, 2, 6, 24, 120, \ldots \quad a_1 = 0, a_n = na_{n-1}$.
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3. $a_{n,k} = a_{n-1,k-1} + a_{n-k,k}, \quad a_{n,0} = 0, a_{n,n} = a_{n,1} = 1 \ n \geq k$
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6. Do you recognize this sequence?
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Remark

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The process of constructing a sequence from a given collection \( \mathbb{C} \), that is building a bijection between \( \mathbb{Z}^+ \) and \( \mathbb{C} \) is called \textbf{enumeration} or \textbf{sequencing}.

Example

\((0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (0, 3), (1, 2), (2, 1), (3, 0), \ldots\) is an enumeration of \( \mathbb{N} \times \mathbb{N} \).
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6. There are many other “named sequences”. We shall study some of them.
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6. There are many other "named sequences". We shall study some of them.

7. We shall start by examining a number of examples.
Examples

For the following sequences try to find a “simple” explicit rule:

1. 1.0.1.0.1.0.... \( a_n = ? \)

Remark
Consider the last example. It was not too difficult to see that \( a_n = 3n - 2 \).
You are probably still struggling with the sequence preceding it. Do you see any relation between it and the last sequence? Can you see it now once your attention was called to it?
Examples

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1. 1.0.1.0.1.0. . . .  \(a_n=?\)
2. 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, . . .  \(a_n=?\)

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3. 3, 6, 11, 18, 27, 38, 51, . . . \( a_n =? \)

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3. 3, 6, 11, 18, 27, 38, 51, …  $a_n =$?
4. 2, 4, 16, 256, 65536, 4294967296, …  $a_n =$?

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5. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, . . .  \( a_n = ? \)
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5. 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, . . . \(a_n = ?\)
6. 1, 2, 4, 8, 14, 25, 45, 79, 138, 240, . . . \(a_n = ?\)

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7. 1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025 . . . \( a_n = ? \)

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Can you see it now once your attention was called to it?
In general, if a finite number of terms of a given sequence are given, we may have multiple rules that will conform to the given data.

**Example**

*What are the rules that will satisfy the sequence 1, 2, 4, ...?*

*Here are a few rules:*

1. \(a_n = 2^n - 1\)
2. \(a_n = \left(\frac{n^2}{2}\right) + 1\)
3. \(a_n = \text{the (smallest prime > } n) - 1\)

**Question**

So which one is the "correct" answer?

**Answer**

All three are correct.

We can find a polynomial \(p(x)\) of degree 2 such that \(p(1) = 1\), \(p(2) = 2\), \(p(3) = 4\).
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*We can find a polynomial \( p(x) \) of degree 2 such that \( p(1) = 1, \ p(2) = 2, \ p(3) = 4. \)
Common Sequences

1. Arithmetic progression: \( a_n = \alpha + (n - 1)d, \quad (a_n - a_{n-1} = d) \)
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3. Binomial coefficients: \( a_{n,k} = \binom{n}{k} \)
4. 2, 3, 5, 7, 11, ... the prime numbers.
5. Given a sequence \( a_n \) define a new sequence: \( s_n = \sum_{k=1}^{n} a_k \).
Let $a_n = \alpha + (n - 1)d$, $s_n = \sum_{i=1}^{n} a_n$. What is $s_n$?
Question

Let \( a_n = \alpha + (n - 1)d \), \( s_n = \sum_{i=1}^{n} a_n \).
What is \( s_n \)?
Sums

**Question**

Let $a_n = \alpha + (n - 1)d$, $s_n = \sum_{i=1}^{n} a_n$.

What is $s_n$?

Let $b_n = \alpha \cdot q^{n-1}$, $S_n = \sum_{i=1}^{n} b_i$.

What is $S_n$?

1. Let $\{a_n\} = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots\}$. 
## Sums

### Question

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**What is** \( S_n \)?

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2. **What is** \( a_n \)?
Question

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5. An interesting sequence: (it has a limit!) $\gamma_n = \log n - \sum_{i=1}^{n} \frac{1}{i}$
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You will see many more sequences throughout this class and in many other classes.
Definition

Let \( n_1 < n_2 < \ldots n_k \subset \mathbb{N} \). \( a_{n_1}, a_{n_2}, \ldots, a_{n_k} \) is a subsequence of the sequence \((a_i)\).
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Example

Let $(a_n) = 2, 5, 10, 17, \ldots n^2 + 1$. The sequence $2, 5, 17, 37$ is a subsequence of $(a_n)$ of length 4.

$n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 6$. 

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A sequence \((a_i)\) is monotonically increasing if \(a_{i+1} > a_i\). (Monotonically, decreasing \((<)\), non-decreasing \((\geq)\), non-increasing \((\leq)\) are defined similarly).
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Question

A little puzzle: 10 policemen stand in a line. Can you prove that there are at least four policemen whose heights are monotonic?
Binary Sequences

Binary sequences play important roles in many areas such as electronics, medicine, economics, engineering, computer science and of course mathematics.

1. Integers have a binary representation.

2. There are $2^n$ distinct binary sequences of length $n$.

3. In many applications we look for binary sequences with particular properties.

Example 1: How many binary sequences of length $2^n$ have exactly $n$ 0's?

Example 2: How many binary sequences of length $n$ do not contain the pattern 010?

Example 3: Can you construct a circular binary sequence of length 32 so that each binary sequence of length 5 is a segment of it? (01001 is a segment of 1001001101).
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