1. Consider a particle of charge $e$ traveling in electromagnetic potentials given by $A(r, t) = -\nabla \Lambda(r, t)$, $\phi(r, t) = \frac{1}{\bar{c}} \frac{\partial \Lambda(r, t)}{\partial t}$, where $\Lambda(r, t)$ is an arbitrary, well-behaved function.

(a) Determine the electromagnetic fields described by these potentials.

(b) Show that the wave function of the particle is given by

$$
\psi(r, t) = \exp \left[ -\frac{i e}{\hbar} \Lambda(r, t) \right] \psi^{(0)}(r, t),
$$

where $\psi^{(0)}$ is the solution to the Schrödinger equation with $\Lambda = 0$.

(c) Let $V(r, t) = e\phi(r, t)$ be a spatially-uniform, time-varying potential. Show that

$$
\psi(r, t) = \exp \left[ -\frac{i e}{\hbar} \int_{-\infty}^{t} \phi(t') dt' \right] \psi^{(0)}(r, t).
$$

Explain the meaning of the lower limit of the integral.

2. Supersymmetric WKB approximation. In lowest order the WKB quantization condition for the one-dimensional potential $V(x)$ is given by

$$
\int_{x_1}^{x_2} \sqrt{2mE_n - V(x)} \, dx = (n + 1/2)\hbar \pi, \quad n = 0, 1, 2, \cdots,
$$

where $x_{1,2}$ are the classical turning points defined by $E_n = V(x_1) = V(x_2)$.

(a) Write this quantization condition for the potential $V_1(x)$ corresponding to the superpotential $W(x)$ in terms of $W(x)$ and the eigenenergies $E_n^{(1)}$. Note that the values of $m$ and $\hbar$ must be kept explicitly here. You may also assume that there is a normalized ground state $\psi^{(1)}_0$. This is said to mean that SUSY is unbroken.

(b) Assume that $W(x)$ is of order $\hbar^0$, Expand your expression from part (a) in powers of $\hbar$ and keep the terms of zeroth and first order in $\hbar$.

(c) Show that the order $\hbar$ term can be written as $\frac{\hbar}{2} \sin^{-1} \left[ \frac{W(x)}{\sqrt{E_n^{(1)}}} \right]^b_a$, where $a$ and $b$ are the classical turning points defined by $E_n^{(1)} = W^2(a) = W^2(b)$ and that the order $\hbar$ term is exactly given by $\hbar \pi / 2$.

(d) Show that the resulting SWKB approximation is given by

$$
\int_a^b \sqrt{2m(E_n^{(1)} - W^2(x))} \, dx = n\hbar \pi, \quad n = 0, 1, 2, \cdots,
$$

The article by Cooper et al. shows examples for which the SWKB approximation works better than the WKB approximation.

3. Continuation of problem 7 from last week. Consider a superpotential $W(x) = \frac{\hbar}{\sqrt{2m}} (x/L)^5$. Use the SWKB approximation to determine the energy levels of the Hamiltonians $H_{1,2}$

4. Degeneracy of one-dimensional Hermitian potentials, $V(x)$. The aim of this problem is to show that for a one-dimensional potential that is finite everywhere, every bound state wave function is non-degenerate and real (except for a possible constant phase factor).

(a) Consider two solutions $u$ and $w$ corresponding to the same energy, $E$. Show that $wu'' - w''u = 0$. and $wu' - uw' = \text{const}$.

(b) Show that $u$ and $w$ obey $u = kw$, where $k$ is a constant (not const of previous equation). This means that $u$ and $w$ are basically the same function. There is only one function that has an eigenvalue $E$, therefore no degeneracies.

(c) Show that $u$ must be real except for a possible constant phase factor.
5. Whereas the Aharonov-Bohm (AB) effect concerns the wave function of a charged particle in the presence of a vector potential \( \mathbf{A} \), the Aharonov-Casher (AC) effect is related to the behavior of a neutral particle with a spin magnetic moment (e.g. a neutron) that moves in an electric field. The AC effect has been demonstrated experimentally by means of a neutron interferometer in a setup shown schematically in the figure.

A neutron wave is split into two coherent parts at \( P \), one of which takes the path \( C \) and the other \( C' \). The spin of the neutron is perpendicular to the page. A very long line charge is normal to the page. The charge per unit length, \( \lambda \), generates an electric field \( \mathbf{E} \). Find a non-relativistic Lagrangian \( L \) that describes the motion of the neutron in this field. The term \( L \) should be linear in \( \mathbf{E} \) and should contain the neutron mass, velocity and spin magnetic moment \( \mu_n \). Show that the phase shift between the two neutron waves at \( Q \) arising from \( \lambda \) is given by

\[
\delta = \pm \frac{\lambda \mu_n}{\hbar c}.
\]

Discuss the signs.

6. Find the Hamiltonian, constants of the motion, energy levels and normalized wave functions of the stationary states of a charged (e), spinless particle moving in a uniform magnetic field \( B \) directed along the \( z \)-axis. Use the vector potential \( A_x = 0, A_y = B x \). You may express your answer in terms of known special functions. Note that this is a three-dimensional problem.

7. Find the Hamiltonian, constants of the motion, energy levels and normalized wave functions of the stationary states of a charged spinless particle moving in a uniform magnetic field \( B \) directed along the \( z \)-axis. Use the vector potential \( \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \). You may express your answer in terms of known special functions. Note that this is a three-dimensional problem.

8. Consider the situation of problems 6 and 7, but also including the effects of a uniform electric field, \( E \), in the \( z \) direction. Determine the energy levels.