Note: This assignment is due on 17 November 2014.

1. (a) Suppose one of the states of an irreducible Markov chain has a self loop. Show that the chain has period 1.
   
   (b) Let \( N \) be a collection of integers closed under addition. Let 100 be the first integer in \( N \) for which the immediate successor is also in \( N \). Find an integer after which every integer is present in \( N \).
   
   (c) Using Euclid’s Algorithm, find \( a, b \) such that \( 100a + 81b = 1 \).

2. At a local 2 year college, \( \frac{2}{3} \) of freshmen become sophomores, \( \frac{1}{4} \) remain freshmen, and \( \frac{1}{12} \) drop out. Two thirds of sophomores graduate, \( \frac{1}{4} \) remain as sophomores and \( \frac{1}{12} \) dropout. Take the states to be ‘F’ for freshmen, ‘S’ for sophomore, ‘D’ for dropout and ‘G’ for graduate. Let \( q(i, R) \) denote the probability that from state \( i \), we will eventually reach the absorbing state \( R \). Let \( l(i, R) \) denote the expected time to reach absorbing state \( R \) from state \( i \).
   
   (a) Write the transition matrix for the Markov chain with rows and columns in the order \( F, S, G, D \).
   
   (b) Write the set of equations for \( q(i, G) \), \( q(i, D) \).
   
   (c) Write the set of equations for \( l(i, G \cup D) \).
   
   (d) What fraction of new students eventually graduate?
   
   (e) What is the expected time of graduation or dropout for a sophomore student?

3. A certain Markov chain has transition matrix

\[
\begin{pmatrix}
A & B & C & D & E \\
A & 0 & 0 & 1/3 & 1/3 & 1/3 \\
B & 1/3 & 1/3 & 1/3 & 0 & 0 \\
C & 0 & 1/3 & 1/3 & 1/3 & 0 \\
D & 0 & 0 & 0 & 0 & 1 \\
E & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(a) Which are the transient and which, the recurrent states? Are there any absorbing states?

(b) Write linear equations for \( q(i, k), l(i, k) \), where \( k \) is an absorbing state.
4. A certain Markov chain has its states partitioned into \( T, R_1, R_2, R_3 \). Let \( x \in T \).

(a) How will you compute the probability of starting from \( x \) and reaching either \( R_1 \) or \( R_2 \)?

(b) Let \( r \in R_1 \). How will you compute the probability of starting from \( x \) and reaching \( r \) before any other recurrent state?

5. Let \( P \) be a square matrix with nonnegative entries. Suppose all rows of \( P \) have row sum strictly less than 1. Show that \( I - P \) is invertible.

6. A certain branching process has the following one step probability for number of progeny:

\[
p_0 = a, p_1 = b, p_2 = c.
\]

For the following cases

- \( a = 1/2, b = 1/4, c = 1/4 \);
- \( a = 1/4, b = 1/4, c = 1/2 \);

(a) Write down the equation the probability of extinction satisfies;

(b) Examine the probability of extinction.