A. Random Variables and the Characteristic Function

- Lecture 1 demonstrates that a large number of random processes (i.e., random jumps) results in a probability distribution that is bell-shaped, i.e., Gaussian. This is a particular case of a more general statement about random processes called the Central Limit Theorem. In this lecture we will prove the Central Limit Theorem.

- Suppose you have an experimental procedure for obtaining a particular result. So, for example, we toss a coin and write 0 for heads (H) and 1 for tails (T). If you cannot predict with certainty the numerical outcome of the procedure $\xi$, then $\xi$ is referred to as a random variable (rv).

- Although we cannot know the results of the procedure in a deterministic way, for lack of initial conditions or because the procedure followed has some inherent complexity that we cannot describe in detail, we can make certain statements about average outcomes after the mapping is performed a large number of times.

- Thus we associate a probability distribution for $\xi$, and refer to the probability that $\xi$ has a value $x$ as $W(x) = W(x)$.

- We have already discussed some of the properties of the probability $W(x)$. Recall $W(x)$ can refer to a discrete or continuous distribution $\sum W(x) = 1 \rightarrow \int dx W(x) = 1$

- Definitions:
  - The average of $\xi$ is defined as $\langle \xi \rangle = \int xW_1(x)dx = \int xW(x)dx$ (2.1)
  - The variance or fluctuation of $x$ is defined as $D(\xi) = \langle (\xi - \langle \xi \rangle)^2 \rangle = \int dx (x - \langle x \rangle)^2 W(x)$ (2.2)
  - The characteristic function $\varphi(a)$ of $W(x)$ is defined as $\varphi(a) = \langle e^{i\alpha\xi} \rangle = \int dx e^{i\alpha x}W(x)$, (2.3)
    which is simply the Fourier transform of $W(x)$.
  - It follows from (2.3) that $W(x) = \frac{1}{\sqrt{2\pi}} \int da e^{-i\alpha x} \varphi(a)$ (2.4)

- A useful expression is the Taylor series expansion of $\ln \varphi(a)$ around $a=0$. This is
  $\ln \varphi(a) = \ln \varphi(0) + a \left( \frac{d \ln \varphi(a)}{da} \right)_{a=0} + \frac{a^2}{2!} \left( \frac{d^2 \ln \varphi(a)}{da^2} \right)_{a=0} + \ldots$ (2.5)
• Straightforward calculations show that
\[
\ln \varphi(0) = 0; \quad \left( \frac{d \ln \varphi(a)}{da} \right)_{a=0} = i \langle \xi \rangle; \quad \left( \frac{d^2 \ln \varphi(a)}{da^2} \right)_{a=0} = -D(\xi)
\]
so that…
\[
\ln \varphi(a) = ia \langle \xi \rangle - \frac{a^2}{2} D(\xi) + \ldots
\]  
(2.7)

• Comment: (2.7) has the general form \( \ln \varphi(a) = \sum_{r=1}^{\infty} \frac{(ia)^r}{r!} K_r \)
where \( K_r \) is called a cumulant.

• Sums of random variables
  - Suppose we have \( n \) random variables. We define \( \xi_n = \sum_{r=1}^{n} \xi_r \) \quad (2.8)
  - For \( n \) statistically independent rv’s the characteristic function is
\[
\varphi(a) = \prod_{r=1}^{n} \varphi_r(a)
\]  
(2.9)

B. The Central Limit Theorem

• The central limit theorem gives the form for the probability distribution for \( n \) statistically independent rv’s \( \{\xi_r\} \) which have identical probability distributions \( W(\xi_r) \), identical averages \( \langle \xi_r \rangle \), and identical variances \( D(\xi_r) \):
\[
\xi_n = \sum_{r=1}^{n} \xi_r; \quad \langle \xi_n \rangle = n \langle \xi_r \rangle; \quad D(\xi_n) = nD(\xi_r)
\]  
(2.10)

• The theorem addresses the limit \( n \to \infty \). The probability distribution for a displacement \( \ell \) after \( n \) jumps where \( n \to \infty \) is …
\[
W(\xi_n = \ell, n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} da \ e^{-ia\ell} \varphi(a) = \frac{1}{2\pi} \int da \ e^{-ia\ell} \exp \left\{ ia \langle \xi_n \rangle - a^2 \frac{D(\xi_n)}{2} + \ldots \right\}
\]
\[
= \frac{1}{2\pi} \int da \ e^{-ia\ell} \exp \left\{ ian \langle \xi_r \rangle - a^2 \frac{D(\xi_r)}{2} + \ldots \right\} \approx \frac{1}{2\pi} \int da \ \exp \left\{ -ia(\ell - \langle \xi_n \rangle) - a^2 \frac{D(\xi_n)}{2} \right\}
\]  
(2.11)

• Suppose \( n \) is large. Then only small values of \( a \) will contribute to the Fourier transform. We truncate the series as shown and set the integration limits to \( \pm \infty \).
Let \( u = i(\ell - \langle \xi_n \rangle) \) then
Equation (2.12) is the Central Limit Theorem. The CLT states that if we have a large number of independent random variables $\xi_n$ with identical averages $\langle \xi \rangle$ and variances $D(\xi)$ the distribution of the sum $\xi_n = \sum_{r=1}^{n} \xi_r$ is a normal distribution (i.e. Gaussian distribution) with average $\langle \xi_n \rangle = n \langle \xi_r \rangle$ and variance $D(\xi_n) = nD(\xi_r)$. The CLT explains the frequent occurrence of normal distributions in nature.