A. The Complex Susceptibility

- In the last lecture we found that the linear response to a force $F(t)$ is given by

$$\Delta B(t) = \int_{-\infty}^{t} dt' F(t') \phi_{BA}(t-t')$$  \hspace{1cm} (19.1)

where the after effect or linear response function is given by

$$\phi_{BA}(t) = \int dX f_{0} \{B(t), A(0)\} = \{B(t), A(0)\}$$  \hspace{1cm} (19.2)

- Assume the applied force has the form

$$F(t) = F_{\omega} \cos \omega t e^{i\omega t}$$  \hspace{1cm} (19.3)

then (19.1) assumes the form...

$$\Delta B(t) = \int_{-\infty}^{t} dt' F(t') \phi_{BA}(t-t') = F_{\omega} \text{Re} \left[ \int_{-\infty}^{t} dt' e^{i\omega t'} \phi_{BA}(t-t') \right]$$  \hspace{1cm} (19.4)

- Now we apply the change of variable $\tau = t-t'$ and (19.4) becomes

$$\Delta B(t) = F_{\omega} \text{Re} \left[ \int_{-\infty}^{t} dt' e^{i\omega t'} \phi_{BA}(t-t') \right] = F_{\omega} \text{Re} \left[ e^{i\omega t} \int_{-\infty}^{t} \tau^{'\omega} e^{-i\omega \tau'} \phi_{BA}(\tau') \right]$$  \hspace{1cm} (19.5)

- Finally (19.5) can be reduced to

$$\Delta B(t) = F_{\omega} \text{Re} \left[ e^{i\omega t} \int_{-\infty}^{t} \tau^{'\omega} e^{-i\omega \tau'} \phi_{BA}(\tau') \right] = F_{\omega} \text{Re} \left[ e^{i\omega t} \chi_{BA}(\omega) \right]$$  \hspace{1cm} (19.6)

where $... \chi_{BA}(\omega) = \int_{0}^{\infty} d\tau e^{-i\omega \tau} \phi_{BA}(\tau)$ is called the complex susceptibility.

- We identify two components of the susceptibility...

$$\chi_{BA}(\omega) = \chi_{BA}'(\omega) - i \chi_{BA}''(\omega)$$

where $... \chi_{BA}'(\omega) = \int_{0}^{\infty} d\tau \phi_{BA}(\tau) \cos \omega \tau$ and $\chi_{BA}''(\omega) = \int_{0}^{\infty} d\tau \phi_{BA}(\tau) \sin \omega \tau$  \hspace{1cm} (19.7)

so that

$$\Delta B(t) = F_{\omega} \left[ \chi_{BA}'(\omega) \cos \omega t + \chi_{BA}''(\omega) \sin \omega t \right]$$  \hspace{1cm} (19.8)
• The term $\chi'_{BA}(\omega)$ is called the in-phase response because it is in phase with the applied force. The term $\chi''_{BA}(\omega)$ is the out-of-phase response.

• There is a relationship between the in-phase $\chi'(\omega)$ and out-of-phase $\chi''(\omega)$ components of the complex susceptibility, called the Kramer-Kronig relations. The derivation of the K.-K. relations is a useful exercise which is applicable to a large number of physical systems.

B. Dissipation

• The physical meaning of the two components of the complex susceptibility $\chi$ will now be discussed. The potential (free) energy of the system in the field of the applied force at time $t$ for $A=B$ and $F(t) = F_0 \cos \omega t$ ...

$$U(t) = -\int F(t) \cdot d[\Delta A(t)]$$

$$= -\int_0^{F_0} F_a \cos \omega t \cdot dF_a \left[ \chi'_{AA} (\omega) \cos \omega t + \chi''_{AA} (\omega) \sin \omega t \right]$$

$$= \left(-\right)\frac{|F_0|^2}{2} \cdot \cos \omega t \left[ \chi'_{AA} (\omega) \cos \omega t + \chi''_{AA} (\omega) \sin \omega t \right]$$

(19.9)

• The average potential (free) energy over one cycle at angular frequency $\omega$ is

$$\bar{U}_\omega = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \cdot U(t)$$

$$= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \cdot \left(-\right)\frac{|F_0|^2}{2} \cdot \cos \omega t \left[ \chi'_{AA} (\omega) \cos \omega t + \chi''_{AA} (\omega) \sin \omega t \right]$$

$$= \left(-\right)\frac{|F_0|^2}{2} \cdot \chi'_{AA} (\omega)$$

(19.10)

• The average rate of energy dissipation over one cycle is

$$\bar{D}_\omega = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt F(t) \left( \frac{d\Delta A}{dt} \right) = \frac{\omega}{2} |F_0|^2 \chi''_{AA} (\omega)$$

(19.11)

C. Currents

• A useful property of Poisson brackets is that

$$\{ f_0, B \} = \frac{f_0}{kT} \{ B, H \} = \frac{f_0 \dot{B}}{kT}$$

(19.12)

Then $\langle [A(0) B(t)] \rangle = \frac{1}{kT} \langle A(0) \dot{B}(t) \rangle = \frac{1}{kT} \langle A(0) B(t) \rangle$

Now we will obtain some expressions useful for calculating a linear current. This means that we assume $B(t) = \dot{A}(t)$
Then the in-phase and out-of-phase components are
\[
\sigma'_{AA}(\omega) = \frac{1}{kT} \int_{0}^{\infty} dt \langle \dot{A}(0) \dot{A}(t) \rangle \cos \omega t \\
\text{and} \quad \sigma''_{AA}(\omega) = \frac{1}{kT} \int_{0}^{\infty} dt \langle \dot{A}(0) \dot{A}(t) \rangle \sin \omega t
\] (19.14)

Note that the average rate of dissipation is given by
\[
\mathcal{D}_\omega = \frac{\omega}{2\pi} \int_{0}^{2\pi/\omega} F(t) d(\Delta A) = \frac{\omega}{2\pi} \int_{0}^{2\pi/\omega} F(t) \frac{d(\Delta A)}{dt} dt = \frac{F_\omega^2}{2} \sigma'_{AA}(\omega)
\] (19.15)

compared to the expression (20.10)
\[
\mathcal{D}_\omega = \omega \frac{|F_\omega|^2}{2} \chi''_{AA}(\omega)
\] (19.16)